SEQUENTIAL SHRINKAGE ESTIMATION

BY

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To Mom

and

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SEQUENTIAL SHRINKAGE ESTIMATION

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This dissertation is concerned with sequential estimation of the multivariate normal mean, estimation of the regression coefficient in a normal linear regression model, and estimation of the difference of mean vectors of two multivariate normal distributions in the presence of unknown and possibly unequal variance-covariance matrices.

For estimating the $p(\geq 3)$ variate normal mean, we consider two different situations. In one case, the covariance matrix is known up to a multiplicative constant; in the other situation, it is entirely unknown but diagonal. In both cases, the sample mean is the maximum likelihood estimator of the population mean. When the covariance matrix is known up to a multiplicative constant, a class of James-Stein estimators is developed which dominates the sample mean under sequential sampling schemes of M. Ghosh, B.K. Sinha, and N. Mukhopadhyay ([1976] Journal of Multivariate Statistics 6, 281-294). Asymptotic risk expansions of the sample mean vector and

James-Stein estimators are provided up to the second order term. Additionally, in this case, some Monte Carlo simulation is done to compare the risks of the sample mean vector, the James-Stein estimators, and a rival class of estimators. In the second case, a class of James-Stein estimators is given which dominates the sample mean asymptotically by considering second order risk expansions.

The next case is concerned with estimation of regression parameters in a Gauss-Markoff setup. Here the classical estimator of the regression coefficient is the least squares estimator, and the sampling scheme used is that of N. Mukhopadhyay ([1974] Journal of the Indian Statistical Association 12, 39-43). Once again, a class of James-Stein estimators that dominates the least squares estimator is developed, and asymptotic risk expansion is given for both the least squares and James-Stein estimators.

Finally, we consider the estimation of the difference of two normal mean vectors, and the sampling schemes developed in 1984 at the Institute of Applied Mathematics National Tsing Hua University by R. Chou and W. Hwang. A class of James-Stein estimators that dominates the difference of sample mean vectors is given. Asymptotic risk expansions are also provided.

CHAPTER ONE
INTRODUCTION

1.1 Sequential Sampling

There are two basic purposes for which sequential methods are used in statistics. The first is to reduce the sample size on an average as compared to the corresponding fixed sample size procedure which meets the same error requirements. Wald's sequential probability ratio test is a classic example of this. The second is to solve certain problems which cannot be solved by any fixed sample procedure. One important example of the latter is the fixed length interval estimation of the normal mean with unknown variance when the confidence coefficient is specified in advance. Another important problem directly related to the topic of this dissertation is a point estimation problem which is described below.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed $N(\theta,\sigma^2)$ variables where $\theta \epsilon R^1$ is unknown. Based on $X_1, \ldots, X_n$, an estimator $\delta_n = \delta_n(X_1,\ldots,X_n)$ is used for estimating θ. Suppose the loss incurred is given by

$$L(\theta,\delta_n) = A|\delta_n - \theta|^2 + cn \quad , \tag{1.1}$$

where $A>0$ is the known weight and $c(>0)$ denotes the known cost per sampling unit. The most natural estimator of θ from a sample of size

n is $\bar{X}_n = \frac{1}{n} \Sigma_{i=1}^{n} X_i$ with associated risk (expected loss)

$$R(\theta,\bar{X}_n) = EL(\theta,\bar{X}_n) = AE|\bar{X}_n - \theta|^2 + cn$$

$$= A\frac{\sigma^2}{n} + cn \ . \qquad (1.2)$$

The above risk is minimized with respect to n at

$$n = n_1^* = [|\ (A\frac{\sigma^2}{c})^{1/2}\ |] \ ,$$

where $[|u|]$ denotes the integer closest to u. Here, for simplicity, we will assume that $(A\frac{\sigma^2}{c})^{1/2}$ is an integer. However, for unknown σ^2 there does not exist any fixed sample size which minimizes (1.2) simultaneously for all σ^2. In this case, motivated from the optimal fixed sample size n_1^* (when σ^2 is known), the following sequential procedure is proposed for determining the sample size.

$$N = \inf\ \{n \geq m\ (\geq 2): n \geq (A\ \tilde{s}_n^2/c)^{1/2}\} \qquad (1.3)$$

where m denotes the initial sample size and

$$\tilde{s}_n^2 = \frac{1}{n-1}\ \Sigma_{i=1}^{n}(X_i - \bar{X}_n)^2 \ , \qquad (1.4)$$

for all $n \geq 2$.

The above sequential procedure was essentially introduced by Robbins (1959) who considered case $L(\theta,\delta_n) = A|\delta_n - \theta| + cn$. Later, Starr (1966) considered the more general case $L(\theta,\delta_n) = A|\delta_n - \theta|^s + cn^t$ with $s>0$ and $t>0$.

The sequential sampling procedure as introduced in (1.3) has sometimes been described as ad hoc. Robbins (1959) did not give any decision theoretic motivation for such a procedure, which as we shall see below can be justified from the minimax criterion.

With this end, assume that σ^2 is known. Suppose θ has a $N(\mu,\tau^2)$ prior. Then the posterior distribution of θ given $X_i = x_i$, $i=1,\ldots,n$, is

$$N\left(\mu + \frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}} (\bar{x}_n - \mu) \ , \ \frac{\sigma^2/n}{1 + \frac{\sigma^2}{n\tau^2}} \right) .$$

Here, under the loss (1.1), the Bayes estimator of θ is

$$\delta_n^B (\bar{X}_n) = \mu + \frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}} (\bar{X}_n - \mu)$$

$$= \mu + \frac{1}{1 + \frac{\sigma^2}{n\tau^2}} (\bar{X}_n - \mu)$$

with corresponding posterior risk equal to $A \dfrac{\sigma^2/n}{1 + (\sigma^2/n\tau^2)} + cn$. The posterior risk being independent of $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$, the

sequential Bayes rule becomes a fixed sample rule with sample size, n_B (determined by minimizing the posterior risk) and estimator $\delta^B_{n_B}(\bar{X}_n)$.

Next, consider a sequence of $\{N(\mu,m),\ m\geq 1\}$ priors. Then the associated sequence of Bayes estimators is $\mu + \frac{1}{1+(\sigma^2/nm)}(\bar{X}_n - \mu)$ with posterior risks $A\frac{\sigma^2/n}{1+(\sigma^2/nm)} + cn$. Since the sequence of posterior risks converges (as $m\to\infty$) to $K = (A\sigma^2/n_1^*) + cn_1^*$ and $R(\theta,\bar{X}_n) = K$ for all θ then the fixed sample rule with sample size determined by minimizing (1.2) with respect to n is a minimax rule.

In summary, n_1^*, the required sample size, and $\bar{X}_{n_1^*}$, the estimator, is a minimax rule. Therefore, one can conceive the stopping rule (1.3) as an empirical minimax stopping rule and not as ad hoc as one might think. (By empirical we mean having any unknowns, in this case σ^2, replaced by their estimators, in this case $\tilde{s}_n^2$).

Next, consider the multivariate analogue to the above problem, i.e., let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ be a sequence of independent and identically distributed $N_p(\underset{\sim}{\theta},\sigma^2\underset{\sim}{V})$ random vectors ($p \geq 3$), with $\theta\epsilon R^p$ unknown and $\underset{\sim}{V}$, positive definite, known. Based on $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$, if $\underset{\sim}{\delta}_n = \underset{\sim}{\delta}_n(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n)$ is used to estimate $\underset{\sim}{\theta}$, suppose the loss incurred is given by

$$L(\underset{\sim}{\theta},\underset{\sim}{\delta}_n) = (\underset{\sim}{\delta}_n - \underset{\sim}{\theta})' \underset{\sim}{Q} (\underset{\sim}{\delta}_n - \underset{\sim}{\theta}) + cn\ , \qquad (1.5)$$

where $\underset{\sim}{Q}$ denotes a known positive definite (weight) matrix, and $c(>0)$ is the known cost per sampling unit. Again, from a sample of size n,

the most natural estimator of $\underset{\sim}{\theta}$ is $\bar{\underset{\sim}{X}}_n = \frac{1}{n}\Sigma_{i=1}^{n}\underset{\sim}{X}_i$, with risk (expected loss)

$$R(\underset{\sim}{\theta},\sigma^2,\bar{\underset{\sim}{X}}_n) = n^{-1}\sigma^2 \text{ tr } (\underset{\sim}{Q}\ \underset{\sim}{V}) + cn \ . \tag{1.6}$$

As before, if σ^2 is known, the above risk is minimized at $n = n_p^* = (\text{tr } \underset{\sim\sim}{QV}/c)^{1/2}$ (here, too, we shall assume that n_p^* is an integer). Once again, if σ^2 is unknown there does not exist any fixed sample size which minimizes (1.6) simultaneously for all σ^2. Again, motivated by the definition of n_p^*, we propose the following sequential sampling scheme.

$$N = \inf\{n \geq m(\geq 2) : n \geq (\tilde{s}_n^2/c)^{1/2}(\text{tr } \underset{\sim}{Q}\ \underset{\sim}{V})^{1/2}\} \tag{1.7}$$

where m denotes the initial sample size and

$$\tilde{s}_n^2 = \{(n-1)p\}^{-1}\sum_{i=1}^{n}(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}_n)'\ \underset{\sim}{V}^{-1}\ (\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}_n) \ , \tag{1.8}$$

for every $n \geq 2$.

As can easily be seen, the above stopping rule is a multivariate generalization of the univariate stopping rule (1.3). Ghosh et al. (1976) and Woodroofe (1977) considered more general stopping rules where Σ was an unknown arbitrary positive definite matrix. To show that the stopping rule (1.7) is more than just an ad hoc extension of (1.3), we will motivate it again from a minimax criterion.

Assume σ^2 is known and suppose $\underset{\sim}{\Theta}$ has a $N(\underset{\sim}{\mu},\tau^2\underset{\sim}{B})$ prior, where $\underset{\sim}{B}$ is a positive definite matrix. Then, for fixed n, the posterior distribution of $\underset{\sim}{\Theta}$ given $\underset{\sim}{X}_i = \underset{\sim}{x}_i$, i=1,...,n, is

$$N(\underset{\sim}{\mu} + \tau^2\underset{\sim}{B}(\tau^2\underset{\sim}{B} + \frac{\sigma^2}{n}\underset{\sim}{V})^{-1}(\bar{\underset{\sim}{x}}_n - \underset{\sim}{\mu}) \ , \ \tau^2\underset{\sim}{B} - \tau^2\underset{\sim}{B}(\tau^2\underset{\sim}{B} + \frac{\sigma^2}{n}\underset{\sim}{V})^{-1}\ \tau^2\underset{\sim}{B}) \ ,$$

with

$$\tau^2\underset{\sim}{B} - \tau^2\underset{\sim}{B}(\tau^2\underset{\sim}{B} + \frac{\sigma^2}{n}\underset{\sim}{V})^{-1}\ \tau^2\underset{\sim}{B} = \frac{\sigma^2}{n}\{\underset{\sim}{V}(\tau^2\underset{\sim}{B} + \frac{\sigma^2}{n}\underset{\sim}{V})^{-1}\ \tau^2\underset{\sim}{B}\}$$

$$= \frac{\sigma^2}{n}\underset{\sim}{V}(\underset{\sim}{I}_p + \frac{\sigma^2}{n\tau^2}\underset{\sim}{B}^{-1}\underset{\sim}{V})^{-1} \ ,$$

where $\underset{\sim}{I}_p$ is the identity matrix of order p. Since the loss (1.5) is quadratic, the Bayes estimator $\underset{\sim}{\delta}_n^B$ of $\underset{\sim}{\Theta}$ is again the mean of its posterior distribution, i.e.,

$$\underset{\sim}{\delta}_n^B(\bar{\underset{\sim}{X}}_n) = \underset{\sim}{\mu} + \tau^2\underset{\sim}{B}(\tau^2\underset{\sim}{B} + \frac{\sigma^2}{n}\underset{\sim}{V})^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu})$$

$$= \underset{\sim}{\mu} + (\underset{\sim}{I}_p + \frac{\sigma^2}{n\tau^2}\underset{\sim}{V}\ \underset{\sim}{B})^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu})$$

with the corresponding posterior risk equal to

$$\frac{\sigma^2}{n}\ \mathrm{tr}[\underset{\sim}{Q}\ \underset{\sim}{V}\ (\underset{\sim}{I}_p + \frac{\sigma}{n\tau^2}\underset{\sim}{B}^{-1}\underset{\sim}{V})^{-1}] + cn \ .$$

Again, the posterior risk is independent of the $\underset{\sim}{X}_i$'s so that the Bayes sequential decision rule is a fixed sample rule with sample size, n_B, determined by minimizing the posterior risk with respect to n.

Next, consider the sequence of $\{N(\underset{\sim}{\mu},m\underset{\sim}{B}),m \geq 1\}$ priors on $\underset{\sim}{\theta}$. Then the Bayes estimator of $\underset{\sim}{\theta}$ becomes

$$\delta_n^B(\underset{\sim}{\bar{X}}_n) = \underset{\sim}{\mu} + (\underset{\sim}{I}_p + \frac{\sigma^2}{nm}\underset{\sim}{V}\underset{\sim}{B})^{-1}(\underset{\sim}{\bar{X}}_n - \underset{\sim}{\mu}) ,$$

with associated posterior risk

$$\frac{\sigma^2}{n}\,\mathrm{tr}[\underset{\sim}{Q}\underset{\sim}{V}(\underset{\sim}{I}_p + \frac{\sigma^2}{nm}\underset{\sim}{B}^{-1}\underset{\sim}{V})^{-1}] + cn .$$

By using a limiting Bayes argument as in the univariate case we see that the posterior risk converges (as $m\to\infty$) to $K = \frac{\sigma^2}{n}\,\mathrm{tr}(\underset{\sim}{Q}\underset{\sim}{V}) + cn$ where $R(\underset{\sim}{\theta},\sigma^2,\underset{\sim}{\bar{X}}_n) = K$ for all $\underset{\sim}{\theta}$. Hence, n_p^*, the sample size, and $\underset{\sim}{\bar{X}}_{n_p^*}$, the estimator, is a minimax rule. Consequently, the proposed stopping rule (1.7) can also be viewed as an empirical minimax stopping rule.

It is the stopping rule (1.7) and its eventual modifications to fit a regression problem, its generalization to a more complex covariance structure and its adaptation to a two sample situation, that will be of interest to us (as far as sequential sampling goes) in the subsequent chapters.

1.2 Shrinkage Estimation

Consider the situation $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ are independent and identically distributed $N_p(\underset{\sim}{\theta}, \sigma^2\underset{\sim}{V})$ random vectors, $\underset{\sim}{\theta} \in R^p$ is unknown, $\underset{\sim}{V}$ is a known positive definite matrix and $p \geq 3$. Again, the problem is to estimate $\underset{\sim}{\theta}$. For a sample of size n, assume the loss (1.5) in estimating $\underset{\sim}{\theta}$ by $\underset{\sim}{\delta}_n = \underset{\sim}{\delta}_n(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$. The minimax estimator of $\underset{\sim}{\theta}$ is $\bar{\underset{\sim}{X}}_n = \frac{1}{n} \Sigma_{i=1}^n \underset{\sim}{X}_i$. However, Stein (1955) was able to establish the existence of estimators of $\underset{\sim}{\theta}$ (for known σ^2 and for $\underset{\sim}{Q} = \underset{\sim}{V} = \underset{\sim}{I}_p$) which dominated $\bar{\underset{\sim}{X}}_n$ (and hence were minimax themselves). Later James and Stein (1961) exhibited some explicit estimators dominating $\bar{\underset{\sim}{X}}_n$. Since then, there have been a multitude of articles concerning extensions of the James-Stein estimators to more general situations. Specifically, we refer to Berger (1976) when σ^2, $\underset{\sim}{Q}$ and $\underset{\sim}{V}$ are known, Efron and Morris (1976) when $\underset{\sim}{Q} = \underset{\sim}{V} = \underset{\sim}{I}_p$ and σ^2 is unknown, and Berger and Bock (1976) when Q is a known diagonal matrix and the covariance matrix is diagonal and unknown. For a general class of estimators that dominate the usual mean vector, we refer to Baranchik (1970), Strawderman (1971), Alam (1973), Efron and Morris (1976), Berger (1976) and Berger (1976).

To motivate the idea of a shrinkage estimator let us consider the case $\underset{\sim}{Q} = \underset{\sim}{V} = \underset{\sim}{I}_p$. Suppose $\underset{\sim}{\theta}$ has a $N(\underset{\sim}{\mu}, \tau^2\underset{\sim}{I}_p)$ prior. Then, by previous results, the Bayes estimator of $\underset{\sim}{\theta}$ given $\underset{\sim}{X}_i = \underset{\sim}{x}_i$, i=1,...,n, is

$$\underset{\sim}{\delta}^B_n(\bar{\underset{\sim}{x}}_n) = \underset{\sim}{\mu} + \frac{\tau^2}{\tau^2 + \frac{\sigma^2}{n}} (\bar{\underset{\sim}{x}}_n - \underset{\sim}{\mu})$$

$$= \underset{\sim}{\mu} + (1 - \frac{\sigma^2/n}{\tau^2 + \frac{\sigma^2}{n}})(\bar{\underset{\sim}{x}}_n - \underset{\sim}{\mu}) \ .$$

Here, if σ^2 and τ^2 are unknown then they must be estimated from the data. With this end, note that marginally

$$\|\bar{\underset{\sim}{x}}_n - \underset{\sim}{\mu}\|^2 \sim (\tau^2 + \frac{\sigma^2}{n}) \ \chi^2_p \ ,$$

where $\|.\|$ denotes the usual Euclidean norm. Hence,

$$E[\frac{\|\bar{\underset{\sim}{x}}_n - \underset{\sim}{\mu}\|^2}{p - 2}]^{-1} = \frac{1}{\tau^2 + \frac{\sigma^2}{n}} \ .$$

Also, marginally

$$\tilde{s}^2_n \sim \frac{\sigma^2}{(n-1)p} \chi^2_{(n-1)p} \ ,$$

where we are referring to definition (1.8) of $\tilde{s}^2_n$. Consequently,

$$E[\tilde{s}^2_n] = \sigma^2 \ .$$

Therefore, replacing the unknowns in $\underset{\sim}{\delta}^{B}(\bar{\underset{\sim}{X}}_n)$ by their estimates we have

$$\underset{\sim}{\delta}_n^{JS} = \underset{\sim}{\mu} + (1 - \frac{(p-2)\tilde{s}_n^2}{n\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\|^2})(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}) ,$$

which is known as the James-Stein estimator. The above empirical Bayes interpretation of James-Steim estimators is due to Efron and Morris (1973). In James and Stein (1961), the estimator was studied from a purely frequentist perspective.

For better understanding of the shrinking property of $\underset{\sim}{\delta}_n^{JS}$, note that

$$\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\| \geq \|\underset{\sim}{\delta}_n^{JS} - \underset{\sim}{\mu}\|$$

if and only if $\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\| \geq |1 - \frac{(p-2)\tilde{s}_n^2}{n\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\|^2}| \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\|$.

Therefore, the distance between $\bar{\underset{\sim}{X}}_n$ and $\underset{\sim}{\mu}$ will be greater than that of $\underset{\sim}{\delta}_n^{JS}$ and $\underset{\sim}{\mu}$ if and only if $1 \geq |1 - \frac{(p-2)\tilde{s}_n^2}{n\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\|^2}|$. When this is true $\underset{\sim}{\delta}_n^{JS}$ will shrink $\bar{\underset{\sim}{X}}_n$ towards the prior mean $\underset{\sim}{\mu}$. To assure that the above condition is <u>always</u> met we sometimes use the plus rule version of $\underset{\sim}{\delta}_n^{JS}$ given by

$$\left(\underset{\sim}{\delta}_n^{JS} \right)^+ = \underset{\sim}{\mu} + \left(1 - \frac{(p-2)\tilde{s}_n^2}{n\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}\|^2}\right)^+ (\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu}) ,$$

where $(a)^+ = \max(0,a)$.

The James-Stein estimator, $\underset{\sim}{\delta}_n^{JS}$, and its eventual modifications to fit a regression problem, its generalizations to a more complex covariance structure and its adaptation to a two-sample problem will be of interest to us (as far as estimation goes) in the subsequent chapters.

1.3 Literature Review

There are only two pertinent articles to date that address the inadmissibility of the sample mean vector, $\bar{\underset{\sim}{X}}_n$, as an estimator of $\underset{\sim}{\theta}$ in a setup where the sample size is random. Ghosh and Sen (1983-84) consider the case of arbitrary $\underset{\sim}{Q}$ and $\underset{\sim}{V}$, but consider a two-stage sampling scheme. The main criticism of this approach is that a two stage procedure requires on an average more observations than n_p^*, the optimal sample size when σ^2 is known (see for example Ghosh and Mukhopadhyay, 1976). Takada (1984), on the other hand, considered the case $\underset{\sim}{Q} = \underset{\sim}{V} = \underset{\sim}{I}$ under a full sequential sampling scheme. The main objection to his approach is that at the n-th stage of sampling his estimator (similar to the James-Stein estimator) uses only p of the available (n-1)p degrees of freedom to estimate σ^2. This will be made clear in Chapter 2.

Before leaving this section let us familiarize ourselves with a particular identity from Stein (1981) which will be used repeatedly in all the proofs. First we require the following lemma.

Lemma 1.1 Suppose $Y \sim N(0,1)$. Then, if $E[|g'(Y)|] < \infty$, one has

$$E[g'(Y)] = E[Yg(Y)]. \tag{1.9}$$

Proof: Write $\phi(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2)$ and note that $\phi'(y) = -y\phi(y)$. Now,

$$E[g'(Y)] = \int_{-\infty}^{\infty} g'(y)\phi(y)dy \tag{1.10}$$

$$= \int_0^{\infty} g'(y)\phi(y)dy + \int_{-\infty}^{0} g'(y)\phi(y)dy$$

$$= \int_0^{\infty} g'(y)[\int_y^{\infty} z\phi(z)dz]dy$$

$$+ \int_{-\infty}^{0} g'(y)[\int_{-\infty}^{y} -z\phi(z)dz]dy$$

$$= \int_0^{\infty} [g(z)-g(0)]z\phi(z)dz + \int_{-\infty}^{0} [g(0)-g(z)](-z\phi(z))dz$$

$$= \int_{-\infty}^{\infty} [g(z)-g(0)]z\phi(z)dz$$

$$= E[Yg(Y)] + g(0)E[Y]$$

$$= E[Yg(Y)] ,$$

since E[Y] = 0. This brings us to the identity, referred to as Stein's identity, which is stated as follows:

Lemma 1.2 Let $X \sim N(\theta,\sigma^2)$. Then if $E_{\theta,\sigma^2}[|h'(X)|] < \infty$, one has

$$E_{\theta,\sigma^2}[h'(X)] = \frac{1}{\sigma^2} E_{\theta,\sigma^2}[(X-\theta)h(X)] . \tag{1.11}$$

Proof: Define g(x) by $g(x) = h(\theta + \sigma x)$. Then, $h(x) = g\left(\frac{X - \theta}{\sigma}\right)$ and $h'(x) = \frac{1}{\sigma} g'\left(\frac{X - \theta}{\sigma}\right)$. Note that $Y = \left(\frac{X - \theta}{\sigma}\right) \sim N(0,1)$. Hence,

$$\begin{aligned} E_{\theta,\sigma^2}[h'(x)] &= \frac{1}{\sigma} E_{\theta,\sigma^2}[g'(\tfrac{X - \theta}{\sigma})] \qquad (1.12)\\ &= \frac{1}{\sigma} E[g'(Y)] \\ &= \frac{1}{\sigma} E[Yg(Y)] \\ &= \frac{1}{\sigma} E[\tfrac{X - \theta}{\sigma} g\ (\tfrac{X - \theta}{\sigma})] \\ &= \frac{1}{\sigma^2} E_{\theta,\sigma^2}[(X - \theta)h(X)] , \end{aligned}$$

where in the third equality in the rhs of (1.12) we use Lemma 1.1. In subsequent chapters, however, we shall require a multivariate version of the above identity which may be stated in the following lemma.

Lemma 1.3 Let $\underset{\sim}{X} \sim N(\underset{\sim}{\theta}, \underset{\sim}{I}_p)$. Then if $E_{\underset{\sim}{\theta}}[\,|\frac{\partial h(\underset{\sim}{X})}{\partial X_i}|\,] < \infty$, for $1 \leq i \leq p$, one has

$$E_{\underset{\sim}{\theta}}\left[\frac{\partial h(\underset{\sim}{X})}{\partial X_i}\right] = E_{\underset{\sim}{\theta}}\left[(X_i - \theta_i)h(\underset{\sim}{X})\right], \tag{1.13}$$

for $1 \leq i \leq p$.

Proof: Note that for each $1 \leq i \leq p$,

$$\begin{aligned} E_{\underset{\sim}{\theta}}\left[\frac{\partial h(\underset{\sim}{X})}{\partial X_i}\right] &= E_{\underset{\sim}{\theta}} E_{\theta_i}\left[\frac{\partial h(\underset{\sim}{X})}{\partial X_i} \mid X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_p\right] \\ &= E_{\underset{\sim}{\theta}} E_{\theta_i}\left[(X_i-\theta_i)h(\underset{\sim}{X}) \mid X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_p\right] \\ &= E_{\underset{\sim}{\theta}}\left[(X_i-\theta_i)h(\underset{\sim}{X})\right], \end{aligned} \tag{1.14}$$

where for the second equality we use Lemma 1.2 and the independence of the X_i's.

An interesting consequence of Lemma 1.3 is the following: If $\underset{\sim}{\delta}(\underset{\sim}{X}) = (X_1 + \phi_1(\underset{\sim}{X}),\ldots, X_p + \phi_p(\underset{\sim}{X}))$ is a rival estimator of $\underset{\sim}{X} \sim N(\underset{\sim}{\theta}, \underset{\sim}{I}_p)$ for estimating $\underset{\sim}{\theta}$, then assuming the loss (1.5) with $\underset{\sim}{Q} = \underset{\sim}{I}_p$ and assuming that $E_{\underset{\sim}{\theta}}[|\frac{\partial \phi_i(\underset{\sim}{X})}{\partial X_i}|] < \infty$ for $1 \leq i \leq p$, the risk difference is given by

$$R(\underset{\sim}{\theta},\delta) - R(\underset{\sim}{\theta},\underset{\sim}{X}) \tag{1.15}$$

$$= E_{\underset{\sim}{\theta}}[\textstyle\sum_{i=1}^{p}(X_i + \phi_i(\underset{\sim}{X}) - \theta_i)^2 - \sum_{i=1}^{p}(X_i - \theta_i)^2]$$

$$= E_{\underset{\sim}{\theta}}[2\textstyle\sum_{i=1}^{p}\phi_i(\underset{\sim}{X})(X_i - \theta_i) + \sum_{i=1}^{p}\phi_i^2(\underset{\sim}{X})]$$

$$= E_{\underset{\sim}{\theta}}[2\textstyle\sum_{i=1}^{p}\dfrac{\partial\phi_i(\underset{\sim}{X})}{\partial X_i} + \sum_{i=1}^{p}\phi_i^2(\underset{\sim}{X})] \ ,$$

where the last equality is obtained from Lemma (1.3). Hence to obtain estimators δ which dominate the usual estimator $\underset{\sim}{X}$ one needs to find solutions to the differential inequality

$$\Delta(\underset{\sim}{x}) \equiv 2\textstyle\sum_{i=1}^{p}\dfrac{\partial\phi_i(\underset{\sim}{x})}{\partial x_i} + \sum_{i=1}^{p}\phi_i^2(\underset{\sim}{x}) < 0 \ . \tag{1.16}$$

Put $\phi_i(\underset{\sim}{x}) = -a(x_i - \mu_i)/\|\underset{\sim}{x} - \underset{\sim}{\mu}\|^2$, the left hand side of (1.16) reduces to $-\dfrac{a(2(p-2)-a)}{\|\underset{\sim}{x} - \underset{\sim}{\mu}\|^2} < 0$ if $0 < a < 2(p-2)$, $p \geq 3$. Hence, for an estimator of the form $\underset{\sim}{X} - \dfrac{a(\underset{\sim}{X} - \underset{\sim}{\mu})}{\|\underset{\sim}{X} - \underset{\sim}{\mu}\|^2}$ dominates $\underset{\sim}{X}$ under the loss (1.5) with $\underset{\sim}{Q} = \underset{\sim}{I}_p$. This is the classical result of James and Stein (1961).

1.4 The Subject of This Research

In Chapter 2, we consider the case of arbitrary $\underset{\sim}{Q}$ and $\underset{\sim}{V}$ (and the loss (1.5)) under a full sequential sampling scheme. We show the exact dominance of a class of estimators over the usual mean vector in estimating $\underset{\sim}{\theta}$. Further, we obtain asymptotic risk expansions for such estimators up to the second order term. A Monte Carlo study is also performed to compare the risks of the sample mean and two rival classes of estimators.

In Chapter 3 we consider a multiple regression set up. Here we wish to estimate the vector of regression parameters $\underset{\sim}{\beta}$ under the loss (1.5) with $\underset{\sim}{Q} = \underset{\sim}{Z}_n' \underset{\sim}{Z}_n$, $\underset{\sim}{Z}_n$ being the usual design matrix of order nxp, and $\underset{\sim}{V} = (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1}$. Again, the sampling is done sequentially with each additional point consisting of p non-stochiastic regressor variables and observing the response (a scalar). Here we show the exact dominance of a class of estimators over the usual least squares estimator. Further, we obtain asymptotic risk expansions for such estimators up to the second order term.

In Chapter 4 we consider a multivariate version of the Behrens-Fisher problem of estimating the difference of two normal means when the variances are assumed to be unequal. The sequential analogue in a univariate setting was first introduced by Ghosh and Mukhopadhyay (1980), and was later generalized to a multivariate setting by Chou and Hwang (1984). We have proposed in Chapter 4 an extension of the stopping rule (1.7) to handle the two sample situation. Also, the

loss (1.5) is used in the estimation of $\underset{\sim}{\mu}$, the difference of two normal mean vectors. We show the dominance of a class of estimators over the difference of the sample mean vectors. Once more we produce an asymptotic risk expansion for such estimators up to the second order term.

In Chapter 5 we return to the one sample problem where $\underset{\sim}{Q}$ is diagonal and the entire covariance matrix, formerly $\sigma^2\underset{\sim}{V}$, is assumed to be diagonal and unknown. With the sequential sampling rule (1.7) generalized to this situation and the loss (1.5), we have produced an asymptotic risk expansion (up to the second order term) for a class of estimators dominating asymptotically the usual mean vector.

CHAPTER TWO
SEQUENTIAL SHRINKAGE ESTIMATION OF THE MEAN OF A MULTIVARIATE NORMAL DISTRIBUTION

2.1 Introduction

Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ be a sequence of independent and identically distributed $N(\underset{\sim}{\theta}, \sigma^2\underset{\sim}{V})$ random vectors, where $\underset{\sim}{\theta} \in R^p$ is unknown, and $\underset{\sim}{V}$, positive definite, is known. The problem is estimation of $\underset{\sim}{\theta}$. Based on $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$, if we estimate $\underset{\sim}{\theta}$ with $\underset{\sim}{\delta}_n = \underset{\sim}{\delta}_n(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n)$, let the loss incurred be (1.5). We have seen from Section (1.2) that for this situation the sequential sampling rule (1.7) and the estimator $\bar{\underset{\sim}{X}}_N$ can be considered as an empirical minimax rule. Consequently, producing a class of estimators which dominate $\bar{\underset{\sim}{X}}_N$ takes on a greater meaning than a mere pathology.

In Section 2.2 we exhibit a class of James-Stein estimators which dominate $\bar{\underset{\sim}{X}}_N$, the usual mean vector, and simultaneously, address the two objections raised in Section 1.3. The first objection was in reference to Ghosh and Sen (1983-84) and their two stage sampling scheme. Unlike a two stage procedure which uses only s_m^2 in defining the stopping rule, the stopping rule (1.7) uses an updated estimator of σ^2 at each stage of the experiment, and thereby demands fewer observations on an average. The second objection was in reference to Takada (1984) and his version of the James-Stein estimator. We shall

see later in this section that Takada's class of estimators use only p of the (n-1)p degrees of freedom available at stage n to estimate σ^2. The class of James-Stein estimators we produce use all the available degrees of freedom for estimating σ^2, and as a consequence allows the estimator to stabilize for large n. In Section 2.3 we produce an asymptotic risk expansion (as $c \to 0$) for the proposed class of estimators. In Section 2.4 we give the results of Monte Carlo simulations comparing the risks of the sample mean, the proposed James-Stein estimators and Takada's estimators. As expected, our estimators achieve greater risk reduction than Takada's.

2.2 A Class of James-Stein Estimators Dominating $\bar{\underset{\sim}{X}}_N$

In this section, we consider the class of James-Stein estimators $\underset{\sim}{\delta}_N^b(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_N)$, where

$$\underset{\sim}{\delta}_n^b(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n) = \bar{\underset{\sim}{X}}_n - \frac{bs_n^2}{n(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda})} \underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}), \quad (2.1)$$

for every $n \geq 2$, where $s_n^2 = \frac{p}{p+2}\tilde{s}_n^2 (n \geq 2)$, b is a constant and $\underset{\sim}{\lambda} \in R^p$ is the known point towards which we want to shrink. Very often $\underset{\sim}{\lambda}$ is taken as the prior mean. The main result of this section is as follows:

Theorem 2.1. Under the stopping rule (1.7), and the loss (1.5),

$$R(\underset{\sim}{\theta},\sigma^2,\delta_{\underset{\sim}{N}}^{b}) < R(\underset{\sim}{\theta},\sigma^2,\bar{\underset{\sim}{X}}_N)$$

for every $b\in(0,2(p-2))$.

Proof: First write

$$R(\underset{\sim}{\theta},\sigma^2,\delta_{\underset{\sim}{N}}^{b}) - R(\underset{\sim}{\theta},\sigma^2,\bar{\underset{\sim}{X}}_N) = -2bE_{\underset{\sim}{\theta},\sigma^2}\left[\frac{s_N^2(\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}(\bar{\underset{\sim}{X}}_N-\underset{\sim}{\theta})}{N(\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda})}\right] \tag{2.2}$$

$$+ b^2E_{\underset{\sim}{\theta},\sigma^2}[s_N^4/(N^2\{(\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda})\})].$$

Since $\underset{\sim}{Q}$ and $\underset{\sim}{V}$ are both positive definite, using the simultaneous diagonalization theorem, there exists a nonsingular $\underset{\sim}{D}$ such that $\underset{\sim}{DQ}^{-1}\underset{\sim}{D}' = \underset{\sim}{I}_p$ and $\underset{\sim}{DVD}' = \underset{\sim}{A}$, a diagonal matrix with all positive diagonal elements. We write a_i as the i^{th} diagonal element of $\underset{\sim}{A}$. Use the transformation $\underset{\sim}{Z}_i = \underset{\sim}{D}(\underset{\sim}{X}_i - \underset{\sim}{\lambda})$, $i=1,2,\ldots$ so that the $\underset{\sim}{Z}_i$'s are iid $N(\underset{\sim}{\zeta},\sigma^2\underset{\sim}{A})$, with $\underset{\sim}{\zeta} = \underset{\sim}{D}(\underset{\sim}{\theta} - \underset{\sim}{\lambda})$. Writing $\bar{\underset{\sim}{Z}}_n = n^{-1}\Sigma_{i=1}^{n}\underset{\sim}{Z}_i$ for every $n \geq 1$, one can rewrite

$$s_n^2 = \{(n-1)(p+2)\}^{-1}\Sigma_{i=1}^{n}(\underset{\sim}{Z}_i - \bar{\underset{\sim}{Z}}_n)'\underset{\sim}{A}^{-1}(\underset{\sim}{Z}_i - \bar{\underset{\sim}{Z}}_n),\ n \geq 2\ ; \tag{2.3}$$

$$(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\theta}) = \bar{\underset{\sim}{Z}}_n'\underset{\sim}{A}^{-1}(\bar{\underset{\sim}{Z}}_n - \underset{\sim}{\zeta}),\ n \geq 1\ ; \tag{2.4}$$

$$(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}) = \bar{\underset{\sim}{Z}}_n'\underset{\sim}{A}^{-2}\bar{\underset{\sim}{Z}}_n,\ n \geq 1\ . \tag{2.5}$$

Then from (2.2), one gets

$$R(\underset{\sim}{\theta},\sigma^2,\delta_N^b) - R(\underset{\sim}{\theta},\sigma^2,\bar{X}_N) = -2bE_{\underset{\sim}{\xi},\sigma^2}[(s_N^2/N)(\bar{Z}_N'A^{-1}(\bar{Z}_N - \underset{\sim}{\xi}))/(\bar{Z}_N'A^{-2}\bar{Z}_N)] \tag{2.6}$$

$$+ b^2E_{\underset{\sim}{\xi},\sigma^2}[(s_N^4/N^2)(\bar{Z}_N'A^{-2}\bar{Z}_N)^{-1}] .$$

Next we use the Helmert orthogonal transformation $Y_2 = (Z_1 - Z_2)/\sqrt{2}$, $Y_3 = (Z_1 + Z_2 - 2Z_3)/\sqrt{6}$,..., $Y_n = (Z_1 + \ldots + Z_{n-1} - (n-1)Z_n)/\sqrt{n(n-1)}$, ... to write

$$s_n^2 = \{(n-1)(p+2)\}^{-1} \Sigma_{i=2}^n Y_i'A^{-1}Y_i, \quad n \geq 2 , \tag{2.7}$$

where the Y_i's are iid $N(\underset{\sim}{0},\sigma^2A)$. Denote by B_n the σ-algebra generated by $Y_2, \ldots, Y_n$. Now,

$$E_{\underset{\sim}{\xi},\sigma^2}[(s_N^2/N)(\bar{Z}_N'A^{-1}(\bar{Z}_N - \underset{\sim}{\xi}))/(\bar{Z}_N'A^{-2}\bar{Z}_N)] \tag{2.8}$$

$$= \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\xi},\sigma^2} E_{\underset{\sim}{\xi},\sigma^2}[\{(s_n^2/n)(\bar{Z}_n'A^{-1}(\bar{Z}_n - \underset{\sim}{\xi}))/(\bar{Z}_n'A^{-2}\bar{Z}_n)\}I_{[N=n]}|B_n]$$

$$= \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\xi},\sigma^2}[n^{-1}s_n^2 I_{[N=n]} E_{\underset{\sim}{\xi},\sigma^2}\{(\bar{Z}_n'A^{-1}(\bar{Z}_n - \underset{\sim}{\xi}))/(\bar{Z}_n'A^{-2}\bar{Z}_n)|B_n]$$

$$= \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\zeta},\sigma^2}[n^{-1}s_n^2 I_{[N=n]}\Sigma_{i=1}^{p} a_i^{-1}(\sigma^2/n)a_i E_{\underset{\sim}{\zeta},\sigma^2}\{\frac{\partial}{\partial \bar{Z}_{ni}}(\frac{\bar{Z}_{ni}}{\bar{Z}_{\sim n}' A^{-2} \bar{Z}_{\sim n}})|B_n]$$

$$= \sigma^2 \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\zeta},\sigma^2}[n^{-2}s_n^2 I_{[N=n]}(p-2)/(\bar{Z}_{\sim n}' A^{-2} \bar{Z}_{\sim n})] .$$

For the second equality in (2.8), one uses the independence of $\bar{Z}_{\sim n}$ and $(Y_{\sim 2},\ldots,Y_{\sim n})$, while for the third equality, one uses Stein's identity (of Stein, 1981).

Next, note that for every n, $n\bar{Z}_{\sim n}' A^{-2} \bar{Z}_{\sim n}/\sigma^2 \sim \Sigma_{i=1}^{p} W_{ni}/a_i$ where the W_{ni}'s are independent and $W_{ni} \sim X_1^2(\frac{n\zeta_i^2}{2\sigma^2 a_i})$, i = 1,...,p. Hence, using the result (see, e.g., Cressie et al., 1981) that if $P(U > 0) = 1$, then $E(U^{-1}) = \int_0^{\infty} E(\exp(-tU))dt$, one gets

$$E_{\underset{\sim}{\zeta},\sigma^2}[n\bar{Z}_{\sim n}' A^{-2} \bar{Z}_{\sim n}/\sigma^2]^{-1} \tag{2.9}$$

$$= \int_0^{\infty} \prod_{i=1}^{p} E[\exp(-t\, W_{ni}/a_i)]dt$$

$$= \int_0^{\infty} \prod_{i=1}^{p} \{(1 + 2ta_i^{-1})^{-1/2} \exp[-nt\zeta_i^2/(\sigma^2 a_i(a_i + 2t))]\}dt$$

$$= g_p(\Delta_n) \text{ (say)} ,$$

where $\Delta_n \equiv \Delta_n(\underset{\sim}{\zeta},\sigma^2)$. In this section we need only that $g_p(\Delta_n)$ is

nonincreasing in n. Now, using again the independence of $\bar{\underset{\sim}{Z}}_n$ and $(\underset{\sim}{Y}_2,\ldots,\underset{\sim}{Y}_n)$, one gets from (2.6) - (2.9),

$$R(\underset{\sim}{\theta},\sigma^2,\underset{\sim}{\delta}^b_N) - R(\underset{\sim}{\theta},\sigma^2,\bar{\underset{\sim}{X}}_N) \tag{2.10}$$

$$= -b\Sigma_{n=m}^{\infty} g_p(\Delta_n) E_{\sigma^2}[(s_n^2/n)(2(p-2) - \frac{bs_n^2}{\sigma^2})I_{[N=n]}]$$

$$< -2b(p-2)\Sigma_{n=m}^{\infty} g_p(\Delta_n) E_{\sigma^2}[n^{-1}s_n^2(1-\frac{s_n^2}{\sigma^2})I_{[N=n]}] ,$$

where in the last step one uses $0 < b < 2(p-2)$. Also, in the above, E_{σ^2} denotes expectation when the $\underset{\sim}{Y}_i$'s are iid $N(\underset{\sim}{0},\sigma^2\underset{\sim}{A})$. Accordingly, for proving the theorem, it suffices to show that

$$\Sigma_{n=m}^{\infty} g_p(\Delta_n) E_{\sigma^2}[n^{-1}s_n^2(1-s_n^2/\sigma^2)I_{[N=n]}] \geq 0 \text{ for all } \sigma^2 > 0 . \tag{2.11}$$

To prove (2.11), first observe that $tr(\underset{\sim\sim}{QV}) = tr(\underset{\sim\sim}{QD}^{-1}\underset{\sim\sim}{AD}'^{-1}) = tr(\underset{\sim}{D}'^{-1}\underset{\sim\sim}{QD}^{-1}\underset{\sim}{A}) = tr(\underset{\sim}{A})$, since $\underset{\sim\sim}{DQ}^{-1}\underset{\sim}{D}' = \underset{\sim}{I}_p$. Let n_0 denote the smallest integer $\geq m$ such that $p(p+2)^{-1}c\, n_0^2/(tr\ \underset{\sim}{A}) \geq \sigma^2$. Then, we write

lhs of (2.11) (2.12)

$$= \Sigma_{n=m}^{n_o-1} g_p(\Delta_n) E_{\sigma^2}[n^{-1}s_n^2(1 - \frac{s_n^2}{\sigma^2})I_{[N=n]}]$$

$$+ g_p(\Delta_{n_o}) E_{\sigma^2}[n_o^{-1}s_{n_o}^2(1 - \frac{s_{n_o}^2}{\sigma^2})I_{[N\geq n_o]}]$$

$$+ \Sigma_{n=n_o}^{\infty} \{g_p(\Delta_{n+1}) E_{\sigma^2}[(n+1)^{-1}s_{n+1}^2(1 - \frac{s_{n+1}^2}{\sigma^2})I_{[N\geq n+1]}]$$

$$- g_p(\Delta_n) E_{\sigma^2}[n^{-1}s_n^2(1 - \frac{s_n^2}{\sigma^2})I_{[N\geq n+1]}]\} ,$$

where the first term in the rhs of (2.11) should be interpreted as zero if $n_o = m$. Note that for $n \geq n_o$, on the set $[N \geq n+1]$, $s_n^2 \geq p(p+2)^{-1}cn^2/(\text{tr } \underset{\sim}{A}) \geq p(p+2)^{-1}cn_o^2/(\text{tr } \underset{\sim}{A}) \geq \sigma^2$. Accordingly, since $g_p(\Delta_n)$ is nonincreasing in n,

third term in the rhs of (2.12) (2.13)

$$\geq \Sigma_{n=n_o}^{\infty} g_p(\Delta_{n+1}) E_{\sigma^2}[\{(n+1)^{-1}s_{n+1}^2(1 - \frac{s_{n+1}^2}{\sigma^2})$$

$$- n^{-1}s_n^2(1 - \frac{s_n^2}{\sigma^2})\}I_{[N\geq n+1]}] .$$

Note that $I_{[N\geq n+1]}$ is a B_n measurable function. Then using the representation $s_{n+1}^2 = ((n-1)s_n^2 + U_{n+1})/n$, where

$U_{n+1} = (p+2)^{-1} \underset{\sim}{Y}'_{n+1} \underset{\sim}{A}^{-1} \underset{\sim}{Y}_{n+1} \sim \sigma^2 \chi^2_p/(p+2)$ independently of $\underset{\sim}{Y}_2, \ldots, \underset{\sim}{Y}_n$, one gets with probability 1,

$$E_{\sigma^2}[(n+1)^{-1} s^2_{n+1}(1 - \frac{s^2_{n+1}}{\sigma^2}) I_{[N \geq n+1]} | B_n] \tag{2.14}$$

$$= I_{[N \geq n+1]} [\frac{(n-1)s^2_n + \frac{p\sigma^2}{p+2}}{(n+1)n} - \frac{(n-1)^2 s^4_n + 2(n-1)s^2_n \frac{p\sigma^2}{p+2} + \frac{p\sigma^4}{p+2}}{(n+1)n^2\sigma^2}]$$

$$= I_{[N \geq n+1]} n^{-1} s^2_n (1 - \frac{s^2_n}{\sigma^2})$$

$$+ I_{[N \geq n+1]} [(s^4_n/\sigma^2)(\frac{1}{n} - \frac{(n-1)^2}{(n+1)n^2}) + s^2_n(\frac{n-1}{(n+1)n} - \frac{2(n-1)p}{n^2(n+1)(p+2)} - \frac{1}{n})$$

$$+ \frac{p\sigma^2}{p+2} (\frac{1}{(n+1)n} - \frac{1}{(n+1)n^2})]$$

$$= I_{[N \geq n+1]} n^{-1} s^2_n (1 - \frac{s^2_n}{\sigma^2})$$

$$+ I_{[N \geq n+1]} [\frac{s^4_n}{\sigma^2} \cdot \frac{3n-1}{n^2(n+1)} - 2\frac{s^2_n}{n^2(n+1)}(n + \frac{(n-1)p}{p+2}) + \frac{p\sigma^2}{p+2} \cdot \frac{n-1}{n^2(n+1)}].$$

Note that the multiple of $I_{[N \geq n+1]}$ in the extreme rhs of (2.14) is a convex function of s^2_n, where the minimum occurs at

$$s_n^2 = \sigma^2 \frac{n + \frac{(n-1)p}{p+2}}{3n-1} < \sigma^2 . \tag{2.15}$$

Hence, recalling that on the set $[N \geq n+1]$, $s_n^2 \geq \sigma^2$, it follows that

$$\text{2nd term in the extreme right of (2.14)} \tag{2.16}$$

$$\geq I_{[N\geq n+1]}\left[\frac{(3n-1)\sigma^2}{n^2(n+1)} - \frac{2\sigma^2(n + \frac{(n-1)p}{p+2})}{n^2(n+1)} + \frac{\sigma^2(n-1)p}{(p+2)n^2(n+1)}\right]$$

$$= I_{[N\geq n+1]} \frac{\sigma^2}{n^2(n+1)}((n-1)(1 - \frac{p}{p+2})) .$$

Hence, from (2.13), (2.14) and (2.16),

$$\text{third term in the rhs of (2.12)} > 0 . \tag{2.17}$$

Next, note that if $n_o = m$, since $I_{[N\geq m]} = 1$ with probability 1 and $E_{\sigma^2}[s_m^2(1- \frac{s_m^2}{\sigma^2})] = \sigma^2\{\frac{p}{p+2} - \frac{(m-1)p((m-1)p+2)}{(m-1)^2(p+2)^2}\} = \frac{(m-1)p\sigma^2}{(m-1)^2(p+2)^2}\, 2(m-2)$ ≥ 0, rhs of (2.12) > 0. For $n_o > m$ (≥ 2), first note that for $n \leq n_o - 1$, on the set $[N = n]$, $s_n^2 \leq p(p+2)^{-1}cn^2/(\text{tr } \underset{\sim}{A}) \leq p(p+2)^{-1}c(n_o-1)^2/(\text{tr } \underset{\sim}{A}) < \sigma^2$ so that using $g_p(\Delta_n)$ nonincreasing in n,

$$\text{first term in the rhs of (2.12)} \tag{2.18}$$

$$\geq g_p(\Delta_{n_o}) \Sigma_{n=m}^{n_o-1} E_{\sigma^2}[n^{-1}s_n^2(1 - \frac{s_n^2}{\sigma^2})I_{[N=n]}] .$$

In view of (2.11), (2.12), (2.17) and (2.18), for proving the theorem it suffices to show that

$$\sum_{n=m}^{n_o-1} E_{\sigma^2}[n^{-1}s_n^2(1 - \frac{s_n^2}{\sigma^2})I_{[N=n]}] + E_{\sigma^2}[n_o^{-1}s_{n_o}^2 (1 - \frac{s_{n_o}^2}{\sigma^2})I_{[N\geq n_o]}] \geq 0 . \tag{2.19}$$

To prove (2.19), first writing $s_{n_o}^2 = ((n_o-2)s_{n_o-1}^2 + U_{n_o})/(n_o-1)$, one gets

$$E_{\sigma^2}[n_o^{-1}s_{n_o}^2 (1 - \frac{s_{n_o}^2}{\sigma^2})I_{[N\geq n_o]}] \tag{2.20}$$

$$= E_{\sigma^2}E_{\sigma^2}[n_o^{-1}s_{n_o}^2 (1 - \frac{s_{n_o}^2}{\sigma^2})I_{[N\geq n_o]}|B_{n_o-1}]$$

$$= E_{\sigma^2}\{[\frac{(n_o-2)s_{n_o-1}^2 + \frac{p}{p+2}\sigma^2}{n_o(n_o-1)} - \frac{(n_o-2)^2 s_{n_o-1}^4 + 2(n_o-2)s_{n_o-1}^2 \frac{p\sigma^2}{p+2} + \frac{p\sigma^4}{p+2}}{n_o(n_o-1)^2\sigma^2}]$$

$$I_{[N\geq n_o]}\}$$

$$= E_{\sigma^2}[\{a_{n_o}s_{n_o-1}^2 - b_{n_o}(s_{n_o-1}^4/\sigma^2) + \sigma^2 c_{n_o}\} I_{[N\geq n_o]}] ,$$

where

$$a_{n_o} = \frac{(n_o-2)(n_o-1-\frac{2p}{p+2})}{n_o(n_o-1)^2}\,,\quad b_{n_o} = \frac{(n_o-2)^2}{n_o(n_o-1)^2}\,,\quad c_{n_o} = \frac{(n_o-2)p}{n_o(n_o-1)^2(p+2)}\,.$$

(2.21)

Let $g_{n_o} = b_{n_o} - a_{n_o} = \frac{(p-2)(n_o-2)}{n_o(n_o-1)(p+2)}$ (> 0), so that $c_{n_o} - g_{n_o} = 2(n_o-2)/\{n_o(n_o-1)(p+2)\} > 0$. Also, let $d_{n_o} = 2b_{n_o} - a_{n_o} = \frac{2(n_o-2)p}{n_o(n_o-1)(p+2)}$ so that $d_{n_o} \in (0,1)$. Now rewrite

extreme right of (2.20) (2.22)

$$= \frac{1}{n_o-1} E_{\sigma^2}\{[d_{n_o}(s^2_{n_o-1} - \frac{s^4_{n_o-1}}{\sigma^2}) + g_{n_o}(\frac{s^4_{n_o-1}}{\sigma^2} - 2s^2_{n_o-1} + \sigma^2)$$

$$+ c_{n_o} - g_{n_o}]I_{[N\geq n_o]}\}$$

$$\geq d_{n_o} E_{\sigma^2}[\frac{1}{n_o-1}(s^2_{n_o-1} - \frac{s^4_{n_o-1}}{\sigma^2})I_{[N\geq n_o]}]\,.$$

If $E_{\sigma^2}[\frac{1}{n_o-1}(s^2_{n_o-1} - \frac{s^4_{n_o-1}}{\sigma^2})I_{[N\geq n_o]}] \geq 0$, noting again that $s^2_n < \sigma^2$ on the set $[N = n]$ for all $n \leq n_o-1$, one proves (2.19) from (2.20) and (2.22). Otherwise noting that $d_{n_o} < 1$, one gets from (2.19), (2.20) and (2.22) that

$$\text{lhs of (2.19)} \tag{2.23}$$

$$\geq \sum_{n=m}^{n_o-2} E_{\sigma^2}[n^{-1}s_n^2(1 - \frac{s_n^2}{\sigma^2})I_{[N=n]}$$

$$+ E_{\sigma^2}[(n_o-1)^{-1}s_{n_o-1}^2(1 - \frac{s_{n_o-1}^2}{\sigma^2})I_{[N\geq n_o-1]}] .$$

Proceed inductively to get either lhs of (2.19) ≥ 0, or finally end with

$$\text{lhs of (2.19)} \geq E_{\sigma^2}[m^{-1}s_m^2(1 - \frac{s_m^2}{\sigma^2})I_{[N\geq m]}] \geq 0 , \tag{2.24}$$

as calculated earlier.

2.3 Asymptotic Risk Expansion for $\bar{X}_N$ and δ_N^b

First we obtain an asymptotic (as $c \to 0$) expansion for the risk of $\bar{X}_N$. Observe

$$R(\underset{\sim}{\theta}, \sigma^2, \bar{X}_N) = 2cn^* + cE_{\sigma^2}[(N-n^*)^2/N] \tag{2.25}$$

$$= 2c^{1/2}\sigma(\text{tr } \underset{\sim}{Q}\underset{\sim}{V})^{1/2} + cE_{\sigma^2}[(N-n^*)^2/N] .$$

From definition $N \overset{a.s.}{\to} \infty$ as $c \to 0$ so that $\tilde{s}_N^2 \overset{a.s.}{\to} \sigma^2$ as $c \to 0$.

Also, using Anscombe's Theorem, $\sqrt{Np}(\tilde{s}_N^2 - \sigma^2) \xrightarrow{L} N(0,2\sigma^4)$ so that using the delta method, $\sqrt{Np}(\tilde{s}_N - \sigma) \xrightarrow{L} N(0,(1/2)\sigma^2)$.

Next use the inequality

$$(\text{tr } \underset{\sim}{QV}/c)^{1/2} \tilde{s}_N \leq N \leq m + (\text{tr } \underset{\sim}{QV}/c)^{1/2} \tilde{s}_{N-1} . \tag{2.26}$$

Dividing all sides of (2.26) by n^*, letting $c \to 0$, and using $\tilde{s}_N \xrightarrow{a.s.} \sigma$ as $c \to 0$, it follows that $N/n^* \xrightarrow{a.s.} 1$ as $c \to 0$. Thus, $\sqrt{n^*p}(\tilde{s}_N - \sigma) \xrightarrow{L} N(0,(1/2)\sigma^2)$, and also $\sqrt{n^*p}(\tilde{s}_{N-1} - \sigma) \xrightarrow{L} N(0,(1/2)\sigma^2)$.

Again from (2.26), one gets the inequality

$$(n^*)^{1/2}(\tilde{s}_N - \sigma)/\sigma \leq (N-n^*)/(n^*)^{1/2} \leq m/(n^*)^{1/2} + (n^*)^{1/2}(\tilde{s}_{N-1} - \sigma)/\sigma. \tag{2.27}$$

Since $m/(n^*)^{1/2} \to 0$ as $c \to 0$, it follows from (2.27) that $(N-n^*)/(n^*)^{1/2} \xrightarrow{L} N(0, \frac{1}{2p})$ as $c \to 0$ which implies that $(N-n^*)/N^{1/2} \xrightarrow{L} N(0, \frac{1}{2p})$ as $c \to 0$ since $N/n^* \xrightarrow{a.s.} 1$ as $c \to 0$. Also, the uniform integrability of $(N-n^*)^2/N$ for all $c \leq c_0$ (specified) can be proved by repeating essentially the arguments of Ghosh and Mukhopadhyay (1980). Thus $E_{\sigma^2}[(N-n^*)^2/N] \to \frac{1}{2p}$ as $c \to 0$, and it follows now from (2.25) that

$$R(\underset{\sim}{\theta},\sigma^2,\bar{\underset{\sim}{X}}_N) = 2c^{1/2}\sigma(\mathrm{tr}\ \underset{\sim}{QV})^{1/2} + \frac{c}{2p} + o(c), \text{ as } c \to 0 . \tag{2.28}$$

Next, we find the asymptotic risk expansion for $\delta^b_{\underset{\sim}{N}}$. It follows from (2.6) and (2.8) - (2.10) that

$$R(\underset{\sim}{\theta},\sigma^2,\delta^b_{\underset{\sim}{N}}) \tag{2.29}$$

$$= R(\underset{\sim}{\theta},\sigma^2,\bar{\underset{\sim}{X}}_N) - 2b(p-2)E_{\sigma^2}[(s_N^2/N)\int_0^\infty \prod_{i=1}^p (1+2ta_i^{-1})^{-1/2} \exp(-\frac{Nt}{\sigma^2}\Sigma_{i=1}^p \frac{\zeta_i^2}{a_i(a_i+2t)})dt]$$

$$+ b^2\sigma^{-2}E_{\sigma^2}[(s_N^4/N)\int_0^\infty \prod_{i=1}^p (1+2ta_i^{-1})^{-1/2} \exp(-\frac{Nt}{\sigma^2}\Sigma_{i=1}^p \frac{\zeta_i^2}{a_i(a_i+2t)})dt] .$$

As $c \to 0$, $s_N^4 \xrightarrow{a.s.} \sigma^4 \frac{p^2}{(p+2)^2}$ and $Nc^{1/2} \xrightarrow{a.s.} (\mathrm{tr}\ \underset{\sim}{QV})^{1/2}\sigma = (\mathrm{tr}\ \underset{\sim}{A})^{1/2}\sigma = (\Sigma_{i=1}^p a_i)^{1/2}\sigma$. Hence, for $\underset{\sim}{\zeta} \neq \underset{\sim}{0}$, as $c \to 0$,

$$c^{-1}(s_N^4/N)\int_0^\infty \prod_{i=1}^p (1+2ta_i^{-1})^{-1/2} \exp(-\frac{Nt}{\sigma^2}\sum_{i=1}^p \frac{\zeta_i^2}{a_i(a_i+2t)})dt \tag{2.30}$$

$$= (s_N^4/(Nc^{1/2}))\int_0^\infty \prod_{i=1}^p (1+2tc^{1/2}a_i^{-1})^{-1/2}$$

$$\exp(-\frac{Ntc^{1/2}}{\sigma^2}\Sigma_{i=1}^p \frac{\zeta_i^2}{a_i(a_i+2tc^{1/2})})dt$$

$$+ \{\frac{p^2}{(p+2)^2}\,\sigma^4/(\sigma(\operatorname{tr} \underset{\sim}{A})^{1/2})\}\int_0^\infty \exp(-\frac{t}{\sigma}(\operatorname{tr} \underset{\sim}{A})^{1/2}\Sigma_{i=1}^p \zeta_i^2/a_i^2)dt$$

$$= \frac{p^2}{(p+2)^2}\,\sigma^4(\operatorname{tr} \underset{\sim}{A})^{-1/2}(\operatorname{tr} \underset{\sim}{A})^{-1/2}(\Sigma_{i=1}^p \zeta_i^2/a_i)^{-1}$$

$$= \frac{p^2}{(p+2)^2}\,\sigma^4(\operatorname{tr} \underset{\sim}{QV})^{-1}\{(\underset{\sim}{\theta} - \underset{\sim}{\lambda})' \underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\underset{\sim}{\theta} - \underset{\sim}{\lambda})\}^{-1} ,$$

since $\operatorname{tr} \underset{\sim}{A} = \operatorname{tr} \underset{\sim}{QV}$, and

$$\Sigma_{i=1}^p \zeta_i^2/a_i^2 = \underset{\sim}{\zeta}'\underset{\sim}{A}^{-2}\underset{\sim}{\zeta} = (\underset{\sim}{\theta}-\underset{\sim}{\lambda})'\underset{\sim}{D}'(\underset{\sim}{DVD}')^{-1}(\underset{\sim}{DVD}')^{-1}\underset{\sim}{D}(\underset{\sim}{\theta}-\underset{\sim}{\lambda}) \qquad (2.31)$$

$$= (\underset{\sim}{\theta}-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{D}^{-1}\underset{\sim}{D}'^{-1}\underset{\sim}{V}^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\lambda})$$

$$= (\underset{\sim}{\theta}-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\lambda}) .$$

Next we prove the uniform integrability (in $c \le c_0$) of the left hand side of (2.30) when $\underset{\sim}{\zeta} \neq \underset{\sim}{0}$. First note that

$$\text{lhs of (2.30)} \qquad (2.32)$$

$$\le (s_N^4/(Nc))[\int_0^1 \exp(-\frac{Nt}{\sigma^2}\Sigma_{i=1}^p \frac{\zeta_i^2}{a_i(a_i+2)})dt$$

$$+ \exp(-\frac{N}{\sigma^2}\Sigma_{i=1}^p \frac{\zeta_i^2}{a_i(a_i+2)})\int_1^\infty (1 + \frac{2t}{\max a_i})^{-\frac{p}{2}} dt]$$

$$\leq (s_N^4/(Nc))(K/N) = Ks_N^4/(Nc^{1/2})^2 ,$$

where K is a constant not depending on c. In deriving the first inequality of (2.32), we have also used that $t/(a_i + 2t)$ $\uparrow$ in t for all i since $a_i > 0$ (i=1,...,p). Thus, it suffices to prove the uniform integrability of $s_N^4/(Nc^{1/2})^2$ in $c \leq c_o$. First use the inequality

$$E_{\sigma^2}[(s_N^4/(N^2c))I_{[s_N^4/N^2c > d]}] \leq d^{-\delta'} E_{\sigma^2}[\{s_N^4/(N^2c)\}^{1+\delta'}] \qquad (2.33)$$

for some $\delta' \in (0, 1/2)$. Taking δ in $(\delta', 1/2)$, and using Holder's inequality, one gets

$$E_{\sigma^2}[\{s_N^4/(N^2c)\}^{1+\delta'}] \qquad (2.34)$$

$$\leq [E_{\sigma^2}(N^2c)^{-(1+\delta)}]^{(1+\delta')/(1+\delta)} [E_{\sigma^2}(s_N^4)^{\frac{(1+\delta)(1+\delta')}{\delta-\delta'}}]^{(\delta-\delta')/(1+\delta)} .$$

Next observe that for any <u>arbitrary</u> ϵ in (0,1),

$$E_{\sigma^2}[(N^2c)^{-(1+\delta)}] \leq E_{\sigma^2}[(N^2c)^{-(1+\delta)} I_{[N \leq \epsilon n^*]}] + (\epsilon^2 n^{*2} c)^{-(1+\delta)}] \qquad (2.35)$$

$$\leq (m^2c)^{-(1+\delta)} P_{\sigma^2}(N \leq \epsilon n^*) + (\epsilon^2 tr(\underset{\sim}{QV})\sigma^2)^{-(1+\delta)} .$$

To proceed further we shall require the following lemma.

Lemma 2.1 For any $0 < \varepsilon < 1$ and $p \geq 1$, $P_{\sigma^2}(N \leq \varepsilon n^*) = O(c^{(1/2)(m-1)p})$.

Proof: Note that

$$P_{\sigma^2}(N \leq \varepsilon n^*) = \sum_{n=m}^{[\varepsilon n^*]} P_{\sigma^2}(N=n) , \tag{2.36}$$

where $[u]$ denotes the largest integer $\leq u$. Now,

$$P_{\sigma^2}(N=m) = P_{\sigma^2}(\tilde{s}_m^2 \leq c \frac{m^2}{tr(\underset{\sim}{QV})}) \tag{2.37}$$

$$= P(\chi^2_{(m-1)p} \leq \frac{(m-1)p}{\sigma^2} c \frac{m^2}{tr(\underset{\sim}{QV})}) .$$

Using the fact that

$$P(\chi_k^2 \leq d) = \frac{(1/2)^{k/2} d^{k/2} e^{-d/2}}{\Gamma(\frac{k}{2}+1)} + P(\chi^2_{k+2} \leq d) \tag{2.38}$$

$$= O_e(d^{k/2}) + O_e(d^{\frac{k+2}{2}})$$

$$= O_e(d^{k/2}) ,$$

we get

$$P_{\sigma^2}(N=m) = O_e(c^{(1/2)(m-1)p}) . \tag{2.39}$$

Therefore all we need show is that

$$\sum_{n=m+1}^{[\epsilon n^*]} P_{\sigma^2}(N=n) = O(c^{(1/2)(m-1)p}) . \tag{2.40}$$

Now, for $n \geq m+1$,

$$P_{\sigma^2}(N=n) \leq P_{\sigma^2}(\tilde{s}_n^2 \leq c \frac{n^2}{tr(\underset{\sim}{QV})}) \tag{2.41}$$

$$= P(\chi^2_{(n-1)p} \leq \frac{(n-1)p}{\sigma^2} c \frac{n^2}{tr(\underset{\sim}{QV})})$$

$$\leq \inf_{h>0} \exp(h \frac{(n-1)p}{\sigma^2} c \frac{n^2}{tr(\underset{\sim}{QV})}) E[\exp(-h\chi^2_{(n-1)p})]$$

$$= \inf_{h>0} \exp(h \frac{(n-1)p}{\sigma^2} c \frac{n^2}{tr(\underset{\sim}{QV})}) (\frac{1}{1+2h})^{\frac{(n-1)p}{2}}$$

$$= \exp[\frac{(n-1)p}{2}(1 - c \frac{n^2}{tr(\underset{\sim}{QV})} \frac{1}{\sigma^2})](c \frac{n^2}{tr(\underset{\sim}{QV})} \frac{1}{\sigma^2})^{\frac{(n-1)p}{2}}$$

$$= [\exp(1 - c \frac{n^2}{tr(\underset{\sim}{QV})} \frac{1}{\sigma^2})(c \frac{n^2}{tr(\underset{\sim}{QV})} \frac{1}{\sigma^2})]^{\frac{(n-1)p}{2}} ,$$

where for the third inequality in (2.41) we use the fact that $P(U \leq d) = \inf_{h>0} e^{hd} E[e^{-hU}]$. Next observe that for $n \leq [\epsilon n^*]$, $c \frac{n^2}{tr(\underset{\sim}{QV})} \frac{1}{\sigma^2} \leq \epsilon^2 < 1$. Since $x \exp(1-x)$ is increasing in x for $0 < x \leq 1$, one gets

$$\Sigma_{n=m+1}^{[\epsilon n^*]} P_{\sigma^2}(N=n) \leq \Sigma_{n=m+1}^{[\epsilon n^*]} [\exp(1 - c\,\frac{n^2}{tr(\underset{\sim}{QV})}\,\frac{1}{\sigma^2})(c\,\frac{n^2}{tr(\underset{\sim}{QV})}\,\frac{1}{\sigma^2})]^{\frac{(n-1)p}{2}}$$

(2.42)

$$\leq c^{\frac{(m-1)p}{2}} (e\,\frac{1}{tr(\underset{\sim}{QV})}\,\frac{1}{\sigma^2})^{\frac{(m-1)p}{2}} \Sigma_{n=m+1}^{[\epsilon n^*]} n^{(m-1)p} [\exp(1-\epsilon^2)\epsilon^2]^{\frac{(n-m)p}{2}}$$

$$\leq c^{\frac{(m-1)p}{2}} (e\,\frac{1}{tr(\underset{\sim}{QV})}\,\frac{1}{\sigma^2})^{\frac{(m-1)p}{2}} \sum_{n=m+1}^{\infty} n^{(m-1)p} [\exp(1-\epsilon^2)\epsilon^2]^{\frac{(n-m)p}{2}}$$

The series in the extreme rhs of (2.42) is convergent. Therefore

$$\Sigma_{n=m+1}^{[\epsilon n^*]} P_{\sigma^2}(N=n) = O(c^{(1/2)(m-1)p}) \,. \tag{2.43}$$

Hence, for $m \geq 2$, $c \leq c_0 < 1$,

$$\text{rhs of (2.35)} \leq K_0 (m^2)^{-(1+\delta)} c_0^{1/2 - \delta} + (\epsilon^2 tr(\underset{\sim}{QV})\sigma^2)^{-(1+\delta)} \,, \tag{2.44}$$

where $K_0 (> 0)$ is a constant not depending on c. Further, using the backward martingale property of $T_n = \{(p+2)/p\} s_n^2$ and Doob's maximal inequality for submartingales, it follows that for every $r \geq 1$,

$$E_{\sigma^2}(s_N^{2r}) = E_{\sigma^2}[\frac{p}{p+2}\, T_N]^r \leq E_{\sigma^2}(T_N)^r \leq E_{\sigma^2}(\sup_{n \geq m} T_n)^r \leq K_1 E_{\sigma^2}(T_m^r) \,,$$

(2.45)

where K_1 is a constant which only depends on r. Combining (2.33) - (2.35) and (2.44) - (2.45), the uniform integrability (in $c \leq c_0$) of $s_N^4/(N^{1/2}c)^2$ follows. Thus, for $m \geq 2$, $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$ (i.e., $\underset{\sim}{\zeta} \neq \underset{\sim}{0}$), as $c \to 0$,

$$c^{-1}E_{\sigma^2}[(s_N^4/N)\int_0^\infty \prod_{i=1}^p (1+2ta_i^{-1})^{-1/2} \exp(-\frac{Nt}{\sigma^2}\Sigma_{i=1}^p \frac{\zeta_i^2}{a_i(a_i+2t)})dt] \tag{2.46}$$

$$\to \frac{p^2}{(p+2)^2}\sigma^4(\text{tr } \underset{\sim\sim}{QV})^{-1}\{(\underset{\sim}{\theta}-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\lambda})\}^{-1} .$$

Similarly, it can be shown that

$$c^{-1}E_{\sigma^2}[(s_N^4/N)\int_0^\infty \prod_{i=1}^p (1+2ta_i^{-1})^{-1/2} \exp(-\frac{Nt}{\sigma^2}\Sigma_{i=1}^p \frac{\zeta_i^2}{a_i(a_i+2t)})dt] \tag{2.47}$$

$$\to \frac{p}{p+2}\sigma^2(\text{tr } \underset{\sim\sim}{QV})^{-1}\{(\underset{\sim}{\theta}-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\lambda})\}^{-1} ,$$

as $c \to 0$. Hence, from (2.28) - (2.29) and (2.46) - (2.47) it follows that for $m \geq 2$, and $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$, $c \to 0$,

$$R(\underset{\sim}{\theta},\sigma^2,\underset{\sim}{\delta}_N^b) \tag{2.48}$$

$$= 2c^{1/2}\sigma(\text{tr } \underset{\sim\sim}{QV})^{1/2} + \frac{1}{2p}c - cb\frac{p^2}{(p+2)^2}(2(p^2-4)p^{-1}-b)\sigma^2(\text{tr } \underset{\sim\sim}{QV})^{-1}$$

$$\times \{(\underset{\sim}{\theta}-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\lambda})\}^{-1} + o(c) .$$

For $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$, i.e., $\underset{\sim}{\zeta} = \underset{\sim}{0}$, it follows from (2.29) that

$$R(\underset{\sim}{\lambda},\sigma^2,\underset{\sim}{\delta}^b_N) \tag{2.49}$$

$$= R(\underset{\sim}{\lambda},\sigma^2,\underset{\sim}{\bar{X}}_N) - 2b(p-2)E_{\sigma^2}(s_N^2/N)\int_0^\infty \prod_{i=1}^p (1+2ta_i^{-1})^{-1/2}\,dt$$

$$+ b^2\sigma^{-2}E_{\sigma^2}(s_N^4/N)\int_0^\infty \prod_{i=1}^p (1+2ta_i^{-1})^{-1/2}\,dt\ .$$

Note that $\int_0^\infty \prod_{i=1}^p (1+2ta_i^{-1})^{-1/2}\,dt = E(\Sigma_{i=1}^p W_i a_i^{-1})^{-1}$, where the W_i's are iid χ^2_1. Note $s_N^2/(Nc^{1/2}) \xrightarrow{a.s.} \frac{p}{p+2}\sigma^2/(\sigma(\operatorname{tr}\underset{\sim\sim}{QV})^{1/2}) = \frac{p}{p+2}\sigma(\operatorname{tr}\underset{\sim\sim}{QV})^{-1/2}$ and $s_N^4/(Nc^{1/2}) \xrightarrow{a.s.} \frac{p^2}{(p+2)^2}\sigma^4/(\sigma(\operatorname{tr}\underset{\sim\sim}{QV})^{1/2}) = \frac{p^2}{(p+2)^2}\sigma^3(\operatorname{tr}\underset{\sim\sim}{QV})^{-1/2}$ as $c \to 0$. Hence, using the uniform integrability (in $c \le c_0$) of $s_N^2/(Nc^{1/2})$ and $s_N^4/(Nc^{1/2})$ for $m \ge 2$, one gets from (2.28) and (2.49) that for $m \ge 2$, as $c \to 0$,

$$R(\underset{\sim}{\lambda},\sigma^2,\underset{\sim}{\delta}^b_N) = 2c^{1/2}\sigma(\operatorname{tr}\underset{\sim\sim}{QV})^{1/2} \tag{2.50}$$

$$- c^{1/2}\sigma b\frac{p^2}{(p+2)^2}\left(2\frac{p^2-4}{p} - b\right)(\operatorname{tr}\underset{\sim\sim}{QV})^{-1/2}$$

$$\times E(\Sigma_{i=1}^p W_i/a_i)^{-1} + o(c^{1/2})\ .$$

Hence, for $0 < b < 2(p-2)$, $m \geq 2$, asymptotically (as $c \to 0$), for $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$, the percent risk improvement of $\delta^b_{\underset{\sim}{N}}$ over $\bar{\underset{\sim}{X}}_N$ is

$$100\{\tfrac{1}{2} b \frac{p^2}{(p+2)^2}(2(p^2-4)/p - b)\sigma\}(\operatorname{tr} \underset{\sim\sim}{QV})^{-3/2} \qquad (2.51)$$

$$\times \{(\underset{\sim}{\theta}-\underset{\sim}{\lambda})'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}(\underset{\sim}{\theta}-\underset{\sim}{\lambda})\}^{-1}c^{1/2} + o(c^{1/2}) .$$

For $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$, $0 < b < 2(p-2)$, $m \geq 2$, asymptotically (as $c \to 0$) the percent risk improvement of $\delta^b_{\underset{\sim}{N}}$ over $\bar{\underset{\sim}{X}}_N$ is

$$100\{\tfrac{1}{2}(\operatorname{tr} \underset{\sim\sim}{QV})^{-1}\} b \frac{p^2}{(p+2)^2}(2(p^2-4)/p - b)E(\Sigma_{i=1}^{p} W_i/a_i)^{-1} + o(1) . \qquad (2.52)$$

Observe that the dominant term in both (2.51) and (2.52) is maximized when $b = (p^2-4)/p$. Unlike the fixed sample case, the optimal choice of b in the sequential case depends on unknown parameters $\underset{\sim}{\theta}$ and σ^2. From an asymptotic point of view as evident in (2.51) and (2.52) it appears that for small c, $b = (p^2-4)/p$ is the optimal choice, which is different from p-2. We may also note that the a_i's are the eigen-values of $\underset{\sim}{A} = \underset{\sim\sim}{DVD}'$, i.e., the eigen-values of $\underset{\sim}{D}'\underset{\sim\sim}{DV} = \underset{\sim\sim}{QV}$. For $\underset{\sim}{Q} = \underset{\sim}{V} = \underset{\sim}{I}_p$, the expressions given in (2.51) and (2.52) simplify respectively to

$$50b \frac{p^2}{(p+2)^2}(2(p^2-4)/p - b)\sigma p^{-3/2}\{(\underset{\sim}{\theta}-\underset{\sim}{\lambda})'(\underset{\sim}{\theta}-\underset{\sim}{\lambda})\}^{-1}c^{1/2} + o(c^{1/2}) \qquad (2.53)$$

and

$$50b\,\frac{p^2}{(p+2)^2}(2(p^2-4)/p - b)p^{-1}(p-2)^{-1} + o(1)\ . \tag{2.54}$$

Remark. It should be noted that in defining $\delta^b_{\underset{\sim}{N}}$ in (2.1), if one uses $\tilde{s}^2_n$ instead of s^2_n, then carries out the asymptotic calculations in the same way as before, one will find that the optimal choice of $b = p - 2$. Then, however, our method of proof does not give an exact dominance result as in Theorem 2.1.

2.4 A Monte Carlo Study

For simplicity, consider in this section the case when $\underset{\sim}{\lambda} = \underset{\sim}{0}$, $\underset{\sim}{Q} = \underset{\sim}{V} = \underset{\sim}{I}_p$. In this special case, Takada (1984) has shown that if one defines

$$\eta_n = (p+2)^{-1}\underset{\sim}{Y}'_n\,\underset{\sim}{Y}_n,\ n \geq 2\ , \tag{2.55}$$

then the estimator $\delta^b_{\underset{\sim}{N}_*} = \delta^b_{\underset{\sim}{N}_*}(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_N)$, where

$$\delta^b_{\underset{\sim}{n}_*} = \bar{\underset{\sim}{X}}_n - \frac{b\eta_n}{n\bar{\underset{\sim}{X}}'_n\,\bar{\underset{\sim}{X}}_n}\,\bar{\underset{\sim}{X}}_n \tag{2.56}$$

dominates $\bar{\underset{\sim}{X}}_N$ for all $0 < b < 2(p-2)$. The difficulty with the estimator η_n is that even for large n it does not stabilize since for every $n \geq 2$, $\eta_n \sim (p+2)^{-1}\chi^2_p$. In this section, our objective is to

compare the risk performance of $\bar{X}_N$, δ^b_N and $\delta^b_{N_*}$, with b = p-2 and $b = (p^2-4)/p$.

For Monte Carlo simulation, we take p = 3. Also, in this section, for actual risk simulation, we consider δ^{p-2}_N and $\delta^{p-2}_{N_*}$ or $\delta^{(p^2-4)/p}_N$ and $\delta^{(p^2-4)/p}_{N_*}$, where $\delta^b_n = \delta^b_n(X_1,\ldots,X_n) = (1 - \frac{bs^2_n}{n\bar{X}_n'\bar{X}_n})^+ \bar{X}_n$, $n \geq 2$, $\delta^b_{n_*} = \delta^b_{n_*}(X_1,\ldots,X_n) = (1 - \frac{bn_n}{n\bar{X}_n'\bar{X}_n})^+ \bar{X}_n$, $n \geq 2$.

In the above $a^+ = \max(a,0)$. Such plus rule versions of James-Stein estimators prevent overshrinking, and in fixed sample situations, perform better than the usual James-Stein estimators.

To simulate the sequential sampling procedure, and evaluate the estimators under consideration, a large pool of trivariate $N(\theta, I_3)$ variables was generated with $\theta' = (0,0,0)$, $(0, \frac{1}{2}, \frac{1}{2})$, $(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(0,1,1)$, $(1,1,1)$, $(0,\sqrt{2},\sqrt{2})$, $(0,1,2)$. Also, c, the cost per sample unit, is taken as c = .01, .05, .1, .25, .5 and 1. It should be noted that the risk of the estimators δ^b_N and $\delta^b_{N_*}$ depends on θ only through $\|\theta\|$, while $\bar{X}_N$ has a risk which does not depend on $\|\theta\|$. For each fixed θ and c under consideration, from the pool, samples were taken sequentially from the top down until 1000 experiments were completed.

A single experiment would be taking sequential samples from the pool until the stopping criterion was met. At this point, sampling would stop, the number of samples would be recorded and the estimators

$\bar{\underset{\sim}{X}}_N$, $\underset{\sim}{\delta}_N^{p-2}$, $\underset{\sim}{\delta}_{N_*}^{p-2}$, $\underset{\sim}{\delta}_N^{(p^2-4)/p}$, $\underset{\sim}{\delta}_{N_*}^{(p^2-4)/p}$, and their associated losses are computed.

On the completion of 1000 experiments, we compute the average losses for $\bar{\underset{\sim}{X}}_N$, $\underset{\sim}{\delta}_N^{p-2}$, $\underset{\sim}{\delta}_N^{(p^2-4)/p}$, $\underset{\sim}{\delta}_{N_*}^{p-2}$, $\underset{\sim}{\delta}_{N_*}^{(p^2-4)/p}$, and these are the simulated versions of the corresponding risks. Also, at this point, we compute the percentage loss improvements

$$100(L(\underset{\sim}{\theta},\bar{\underset{\sim}{X}}_N) - L(\underset{\sim}{\theta},\underset{\sim}{\delta}_N^{p-2}))/L(\underset{\sim}{\theta},\bar{\underset{\sim}{X}}_N);$$

$$100(L(\underset{\sim}{\theta},\bar{\underset{\sim}{X}}_N) - L(\underset{\sim}{\theta},\underset{\sim}{\delta}_N^{(p^2-4)/p}))/L(\underset{\sim}{\theta},\bar{\underset{\sim}{X}}_N);$$

$$100(L(\underset{\sim}{\theta},\bar{\underset{\sim}{X}}_N) - L(\underset{\sim}{\theta},\underset{\sim}{\delta}_{N_*}^{p-2}))/L(\underset{\sim}{\theta},\bar{\underset{\sim}{X}}_N);$$

$$100(L(\underset{\sim}{\theta},\bar{\underset{\sim}{X}}_N) - L(\underset{\sim}{\theta},\underset{\sim}{\delta}_{N_*}^{(p^2-4)/p}))/L(\underset{\sim}{\theta},\bar{\underset{\sim}{X}}_N) \ .$$

Our simulation findings are summarized in Table 2.1 and Figures 2.1 to 2.6. It is clear from the table that as in the fixed sample case, when $\underset{\sim}{\lambda} = \underset{\sim}{0}$, the risk improvement of all the estimators is most substantial when $\|\underset{\sim}{\theta}\| = 0$, and the improvement keeps diminishing as $\|\underset{\sim}{\theta}\|$ moves further and further away from zero. Also, it is clearly evident that with the proposed stopping rule $b = (p^2-4)/p$ does better than $b = p-2$. This is also clear from the asymptotic risk expansion. Also, for a fixed $\|\underset{\sim}{\theta}\| \neq 0$ as c decreases, i.e., the average sample size gets larger, the percentage risk improvement decreases as in the fixed sample case.

Table 2.1 The Risk and the Percentage Risk Improvements

c	$\|\|\underset{\sim}{\theta}\|\|$	$\bar{N}$	$R(\bar{X}_N, \underset{\sim}{\theta})$	$R(\underset{\sim}{\delta}_N^{p-2}, \underset{\sim}{\theta})$	$R(\underset{\sim}{\delta}_N^{\frac{p^2-4}{p}}, \underset{\sim}{\theta})$	$R(\underset{\sim}{\delta}_{N_*}^{p-2}, \underset{\sim}{\theta})$	$R(\underset{\sim}{\delta}_{N_*}^{\frac{p^2-4}{p}}, \underset{\sim}{\theta})$	$\%\underset{\sim}{\delta}_N^{p-2}$	$\%\underset{\sim}{\delta}_N^{\frac{p^2-4}{p}}$	$\%\underset{\sim}{\delta}_{N_*}^{p-2}$	$\%\underset{\sim}{\delta}_{N_*}^{\frac{p^2-4}{p}}$
1	0	2.27	3.6948	3.3591	3.2005	3.4022	3.2586	9.09	13.38	7.92	11.81
.50	0	2.72	2.5424	2.2831	2.1600	2.3333	2.2286	10.20	15.04	8.22	12.34
.25	0	3.62	1.8166	1.6114	1.5138	1.6681	1.5939	11.30	16.67	8.17	12.26
.10	0	5.54	1.1751	1.0313	.9635	1.0739	1.0240	12.24	18.01	8.61	12.86
.05	0	7.89	.8052	.6959	.6441	.7276	.6897	13.57	20.01	9.64	14.34
.01	0	17.60	.3490	.2975	.2737	.3136	.2969	14.76	21.58	10.14	14.93
1	$\sqrt{1/2}$	2.26	3.6412	3.4018	3.2895	3.4246	3.3197	6.57	9.66	5.95	8.83
.50	$\sqrt{1/2}$	2.73	2.4466	2.2889	2.2154	2.3202	2.2581	6.45	9.45	5.17	7.70
.25	$\sqrt{1/2}$	3.56	1.8518	1.7350	1.6779	1.7614	1.7164	6.31	9.39	4.88	7.31
.10	$\sqrt{1/2}$	5.45	1.1591	1.1020	1.0757	1.1179	1.0976	4.93	7.20	3.55	5.31
.05	$\sqrt{1/2}$	7.82	.8002	.7680	.7550	.7773	.7667	4.02	5.65	2.86	4.19
.01	$\sqrt{1/2}$	17.48	.3694	.3618	.3589	.3642	.3625	2.06	2.84	1.41	1.87
1	1	2.27	3.6677	3.4908	3.4107	3.5091	3.4329	4.82	7.01	4.32	6.40
.50	1	2.78	2.5991	2.4761	2.4223	2.4985	2.4520	4.73	6.80	3.87	5.66
.25	1	3.58	1.8506	1.7787	1.7461	1.7950	1.7677	3.89	5.65	3.00	4.48
.10	1	5.39	1.1681	1.1326	1.1173	1.1427	1.1317	3.04	4.35	2.17	3.12
.05	1	7.83	.8107	.7969	.7926	.7996	.7958	1.70	2.23	1.37	1.84
.01	1	17.58	.3524	.3497	.3493	.3507	.3506	.77	.88	.48	.51

1	$\sqrt{2}$	2.27	3.6316	3.5422	3.5054	3.5470	3.5084	2.46	3.48	2.33	3.39
.50	$\sqrt{2}$	2.74	2.5176	2.4638	2.4438	2.4710	2.4505	2.14	2.93	1.85	2.67
.25	$\sqrt{2}$	3.57	1.7857	1.7611	1.7544	1.7656	1.7590	1.38	1.75	1.13	1.50
.10	$\sqrt{2}$	5.45	1.1648	1.1489	1.1429	1.1528	1.1481	1.37	1.88	1.03	1.43
.05	$\sqrt{2}$	7.74	.7940	.7859	.7832	.7880	.7861	1.02	1.36	.76	.99
.01	$\sqrt{2}$	17.62	.3516	.3502	.3499	.3508	.3508	.40	.48	.23	.23
1	$\sqrt{3}$	2.28	3.6506	3.5938	3.5726	3.5954	3.5710	1.56	2.14	1.51	2.18
.50	$\sqrt{3}$	2.73	2.5321	2.4894	2.4741	2.4966	2.4830	1.69	2.29	1.40	1.94
.25	$\sqrt{3}$	3.59	1.8786	1.8575	1.8497	1.8639	1.8580	1.12	1.54	.78	1.10
.10	$\sqrt{3}$	5.48	1.1659	1.1586	1.1572	1.1601	1.1585	.63	.75	.50	.63
.05	$\sqrt{3}$	7.80	.8181	.8147	.8144	.8154	.8149	.42	.45	.33	.39
.01	$\sqrt{3}$	17.63	.3496	.3486	.3483	.3489	.3488	.29	.37	.20	.23
1	2	2.27	3.6174	3.5750	3.5641	3.5779	3.5660	1.17	1.47	1.09	1.42
.50	2	2.68	2.5962	2.5701	2.5627	3.5713	2.5621	1.01	1.29	.96	1.31
.25	2	3.62	1.7980	1.7867	1.7849	1.7884	1.7859	.63	.73	.53	.67
.10	2	5.46	1.1870	1.1796	1.1170	1.1815	1.1796	.62	.84	.46	.62
.05	2	7.91	.7930	.7883	.7866	.7895	.7883	.59	.81	.44	.59
.01	2	17.63	.3501	.3492	.3490	.3494	.3491	.26	.31	.20	.29

Table 2.1-continued

c	$\|\underset{\sim}{\theta}\|$	$\bar{N}$	$R(\bar{X}_N, \underset{\sim}{\theta})$	$R(\underset{\sim}{\delta}_N^{p-2}, \underset{\sim}{\theta})$	$R(\underset{\sim}{\delta}_N^{\frac{p^2-4}{p}}, \underset{\sim}{\theta})$	$R(\underset{\sim}{\delta}_{N_*}^{p-2}, \underset{\sim}{\theta})$	$R(\underset{\sim}{\delta}_{N_*}^{\frac{p^2-4}{p}}, \underset{\sim}{\theta})$	$\%\underset{\sim}{\delta}_N^{p-2}$	$\%\underset{\sim}{\delta}_N^{\frac{p^2-4}{p}}$	$\%\underset{\sim}{\delta}_{N_*}^{p-2}$	$\%\underset{\sim}{\delta}_{N_*}^{\frac{p^2-4}{p}}$
1	$\sqrt{5}$	2.28	3.5773	3.5465	3.5390	3.5480	3.5394	.86	1.07	.82	1.06
.50	$\sqrt{5}$	2.73	2.5431	2.5247	2.5201	2.5274	2.5221	.72	.90	.62	.83
.25	$\sqrt{5}$	3.53	1.7944	1.7856	1.7842	1.7881	1.7872	.49	.57	.35	.40
.10	$\sqrt{5}$	5.57	1.1658	1.1620	1.1616	1.1634	1.1634	.33	.36	.21	.21
.05	$\sqrt{5}$	7.84	.8193	.8159	.8147	.8169	.8161	.41	.56	.29	.39
.01	$\sqrt{5}$	17.44	.3496	.3491	.3490	.3492	.3491	.14	.17	.11	.14

The opposite is the case when $\|\underset{\sim}{\theta}\| = 0$. The main reason is that when $\underset{\sim}{\lambda} = \underset{\sim}{0}$, $N\|\bar{\underset{\sim}{X}}_N\|^2$ behaves as a multiple of $c^{-1/2}\|\underset{\sim}{\theta}\|^2$ when $\underset{\sim}{\theta} \neq \underset{\sim}{0}$, while $N\|\bar{\underset{\sim}{X}}_N\|^2 \xrightarrow{L} \chi^2_p$ when $\underset{\sim}{\theta} = \underset{\sim}{0}$.

Figure 2.1 plots the risk of the sample mean, $\bar{\underset{\sim}{X}}_N$, the James-Stein estimator at b = p-2, $\underset{\sim}{\delta}_N^{p-2}$, and Takada's estimator at b = p-2, $\underset{\sim}{\delta}_{N_*}^{p-2}$, versus $\|\underset{\sim}{\theta}\|$, for six levels of cost. As can be seen from the graphs, $\underset{\sim}{\delta}_{N_*}^{p-2}$, for each level of cost, has smaller risk than $\bar{\underset{\sim}{X}}_N$ while, as was expected, $\underset{\sim}{\delta}_N^{p-2}$ has smaller risk than $\underset{\sim}{\delta}_{N_*}^{p-2}$. Also, we can see that the risk difference is greatest at $\|\underset{\sim}{\theta}\| = 0$ and diminishes as $\|\underset{\sim}{\theta}\|$ moves away from zero. Figure 2.2 plots the percent risk improvement of $\underset{\sim}{\delta}_N^{p-2}$ over $\bar{\underset{\sim}{X}}_N$, and $\underset{\sim}{\delta}_{N_*}^{p-2}$ over $\bar{\underset{\sim}{X}}_N$, versus cost, for four levels of $\|\underset{\sim}{\theta}\|$. Here we have the percent risk improvement of $\underset{\sim}{\delta}_N^{p-2}$ being better than that of $\underset{\sim}{\delta}_{N_*}^{p-2}$, for each level of $\|\underset{\sim}{\theta}\|$. Also, we see that the percent risk improvement is greatest at $\|\underset{\sim}{\theta}\| = 0$ and increases as c decreases, while for $\|\underset{\sim}{\theta}\| \neq 0$ the percent risk improvement decreases as c decreases. Further, for fixed cost, the percent risk improvement decreases as $\|\underset{\sim}{\theta}\|$ moves away from zero. Figure 2.3 plots the risks of the sample mean, $\bar{\underset{\sim}{X}}_N$, the James-Stein estimator at b = $(p^2-4)/p$, $\underset{\sim}{\delta}_N^{(p^2-4)/p}$, and Takada's estimator at b = $(p^2-4)p$, $\underset{\sim}{\delta}_{N_*}^{(p^2-4)/p}$, versus $\|\underset{\sim}{\theta}\|$, for the six previous levels of cost. The patterns and conclusions for these plots are identical to those of Figure 2.1. Figure 2.4 plots the percent risk improvement of $\underset{\sim}{\delta}_N^{(p^2-4)/p}$ over $\bar{\underset{\sim}{X}}_N$, and $\underset{\sim}{\delta}_{N_*}^{(p^2-4)/p}$ over $\bar{\underset{\sim}{X}}_N$, versus cost, for the four

previous levels of $\|\underset{\sim}{\theta}\|$. The patterns and conclusions for these plots are identical to those of Figure 2.2. Figure 2.5 is a plot of the risks of $\bar{\underset{\sim}{X}}_N$, $\underset{\sim}{\delta}_N^{p-2}$ and $\underset{\sim}{\delta}_N^{(p^2-4)/p}$, versus $\|\underset{\sim}{\theta}\|$, for the six previous levels of cost. This figure gives us an idea of how Figures 2.1 and 2.3 compare as far as the James-Stein estimators are concerned. Here, for the first time, we can graphically see that $\underset{\sim}{\delta}_N^{(p^2-4)/p}$ has smaller risk than $\underset{\sim}{\delta}_N^{(p-2)}$ (while, as before, both dominate $\bar{\underset{\sim}{X}}_N$). Hence, we have more evidence in favor of $(p^2-4)/p$ as the optimal choice of b. Figure 2.6 plots the percent risk improvements of $\underset{\sim}{\delta}_{N_*}^{p-2}$, $\underset{\sim}{\delta}_N^{p-2}$, $\underset{\sim}{\delta}_{N_*}^{(p^2-4)/p}$ and $\underset{\sim}{\delta}_N^{(p^2-4)/p}$, over $\bar{\underset{\sim}{X}}_N$, versus cost, for two levels of $\|\underset{\sim}{\theta}\|$. Here, for the first time, we can graphically see that $\underset{\sim}{\delta}_N^{(p^2-4)/p}$ has the greatest improvement while $\underset{\sim}{\delta}_{N_*}^{(p^2-4)/p}$ is second, $\underset{\sim}{\delta}_N^{p-2}$ is third and $\underset{\sim}{\delta}_{N_*}^{p-2}$ has the least improvement over $\bar{\underset{\sim}{X}}_N$ giving once again more evidence in the favor of $(p^2-4)/p$ as the optimal choice of b.

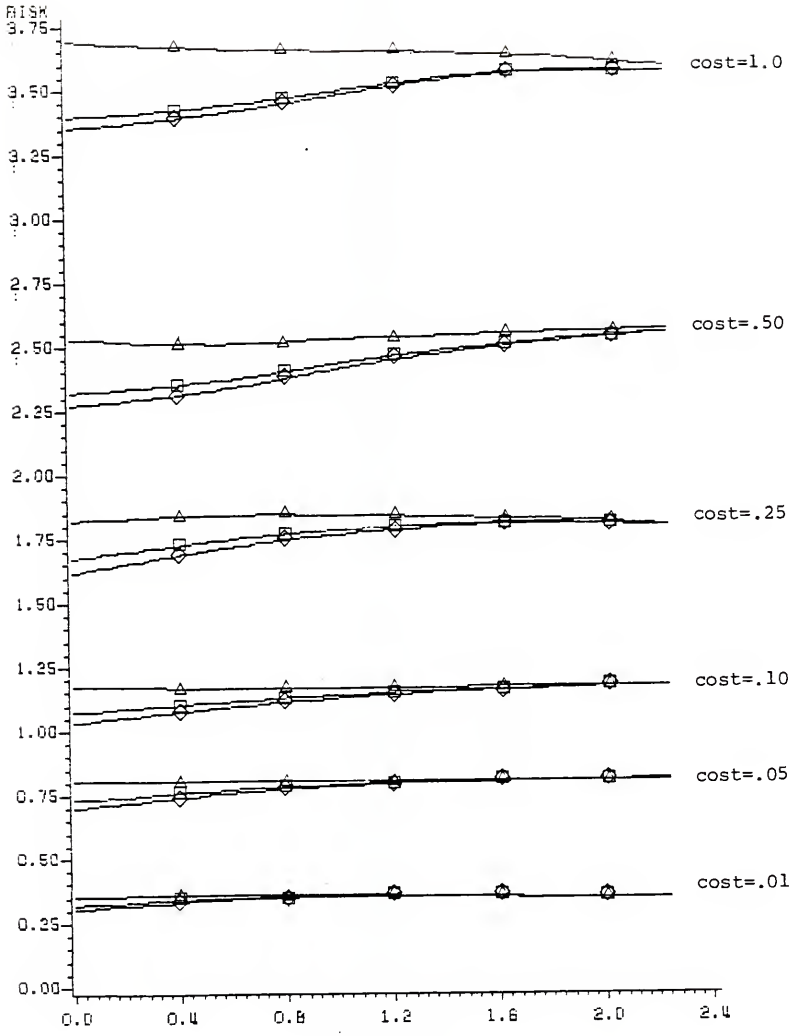
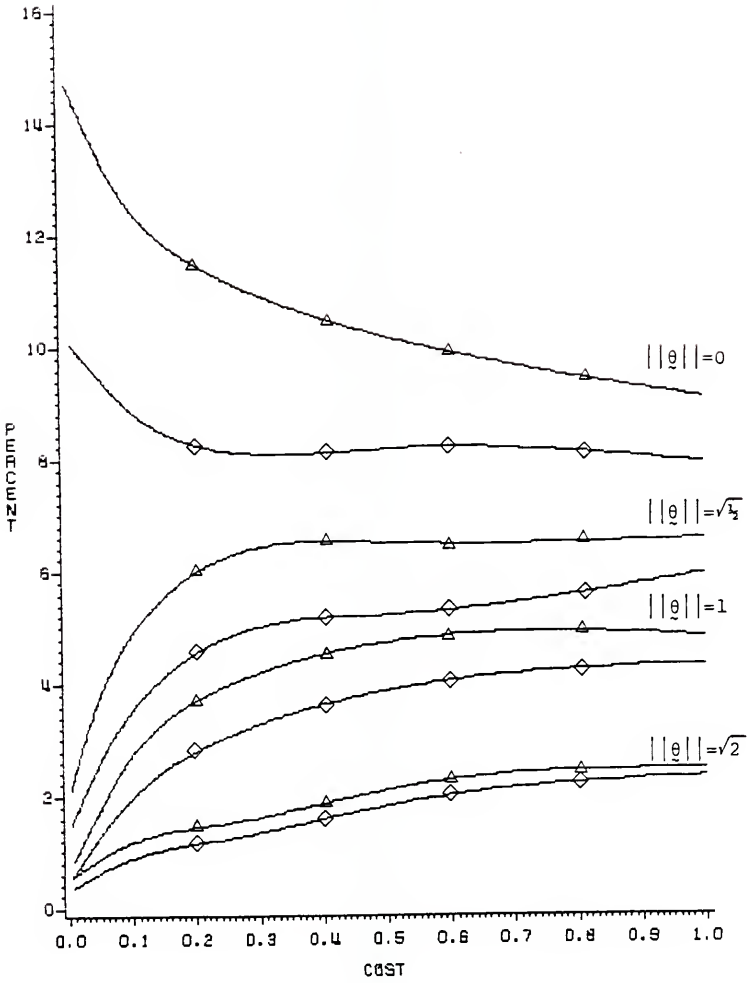


Figure 2.1 Risk Analysis at b=p-2.



△ denotes percent risk improvement for δ_{N}^{p-2}

◇ denotes percent risk improvement for $\delta_{N_*}^{p-2}$

Figure 2.2 Percent Risk Improvement at b=p-2.

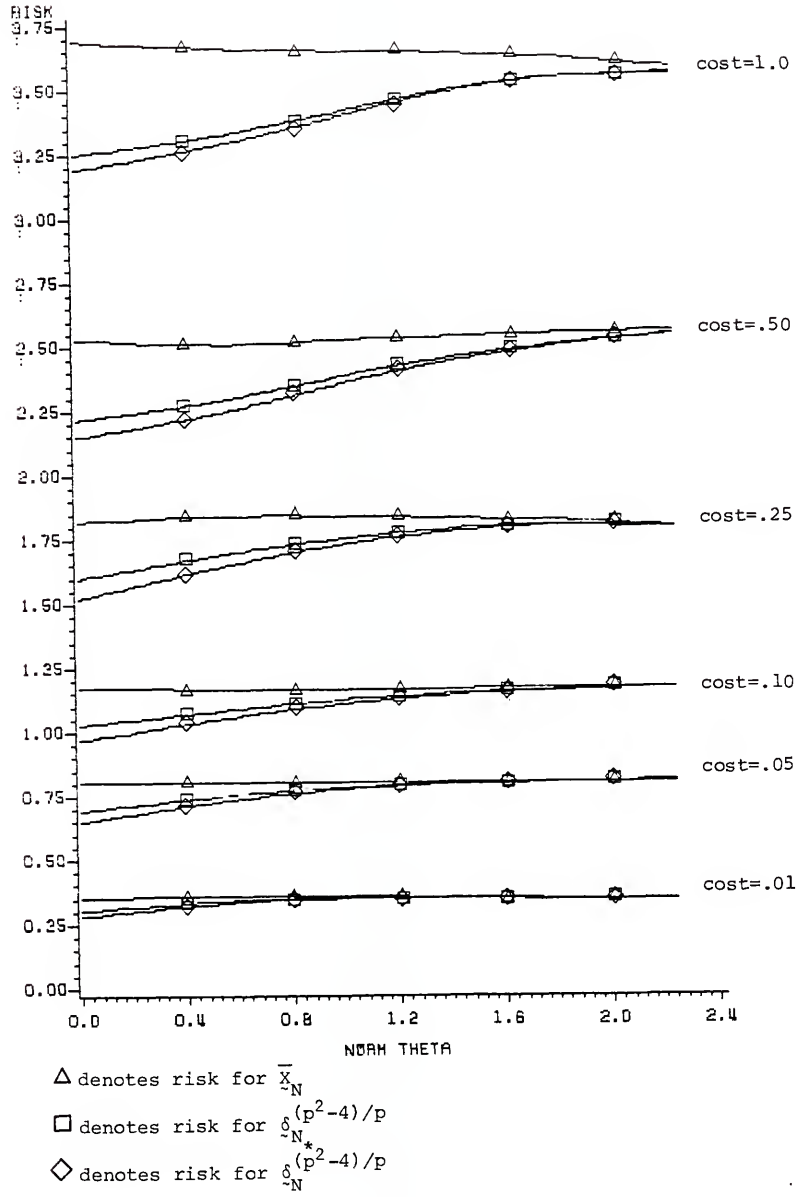
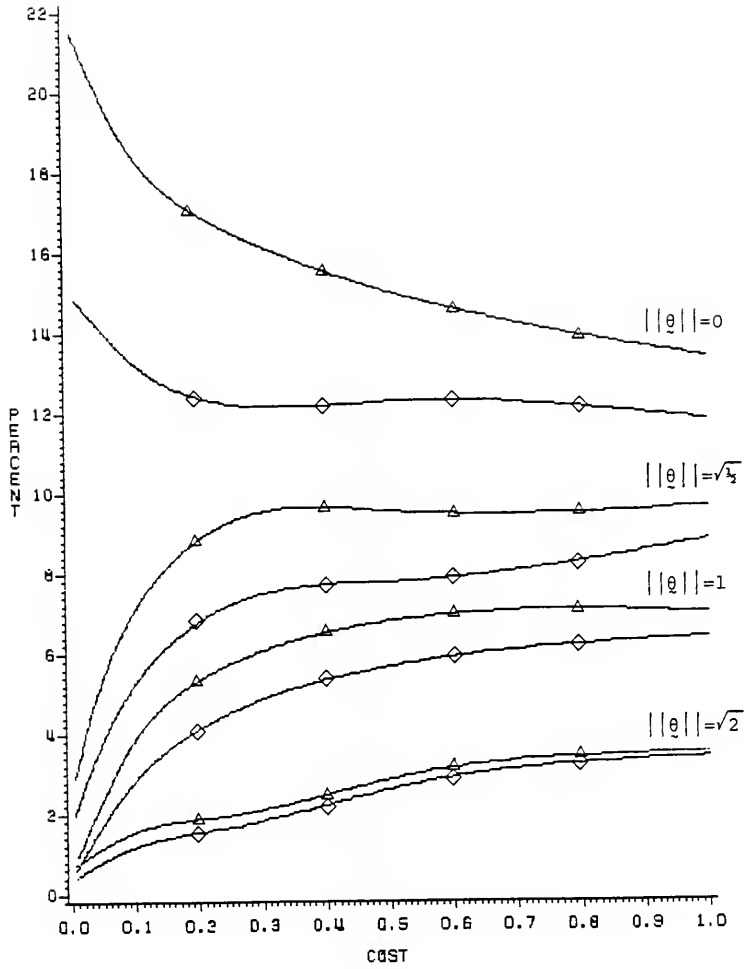


Figure 2.3 Risk Analysis at $b=(p^2-4)/p$.



△ denotes percent risk improvement for $\delta_N^{(p^2-4)/p}$

◇ denotes percent risk improvement for $\delta_{N_*}^{(p^2-4)/p}$

Figure 2.4 Percent Risk Improvement at $b=(p^2-4)/p$.

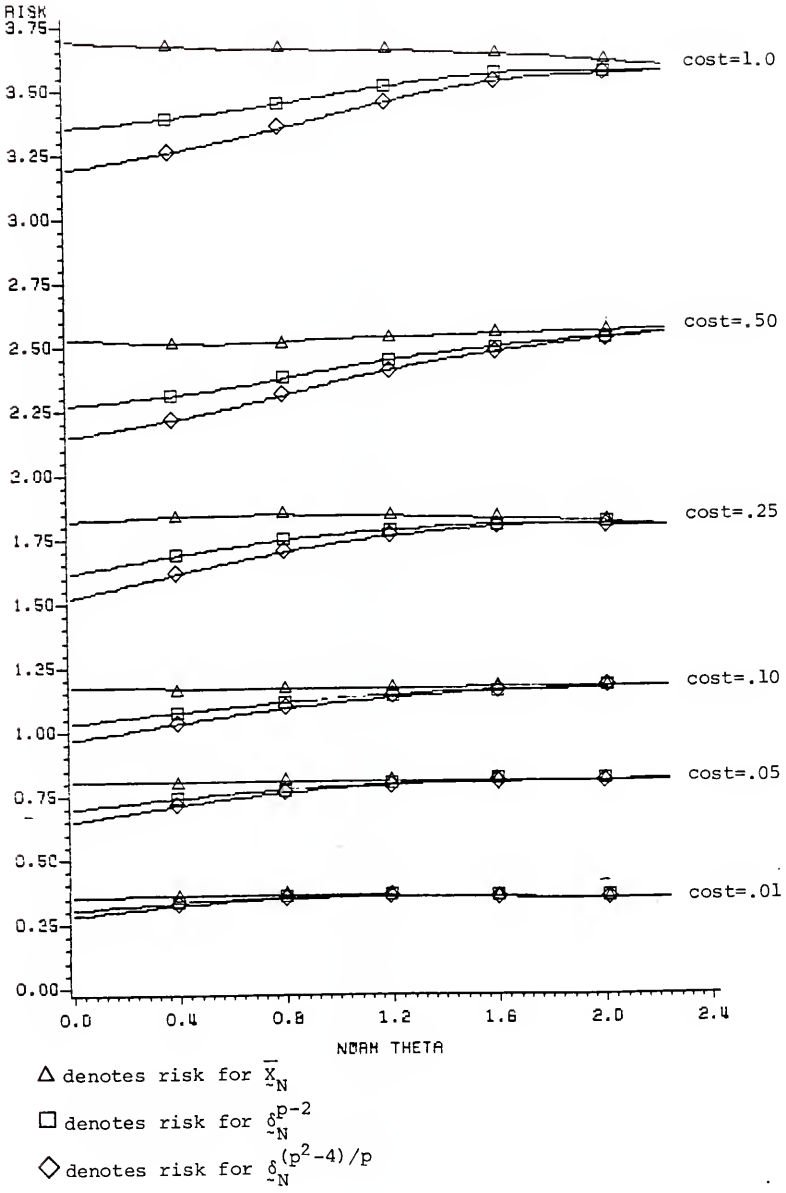


Figure 2.5 Risk Analysis for the James-Stein Estimators.

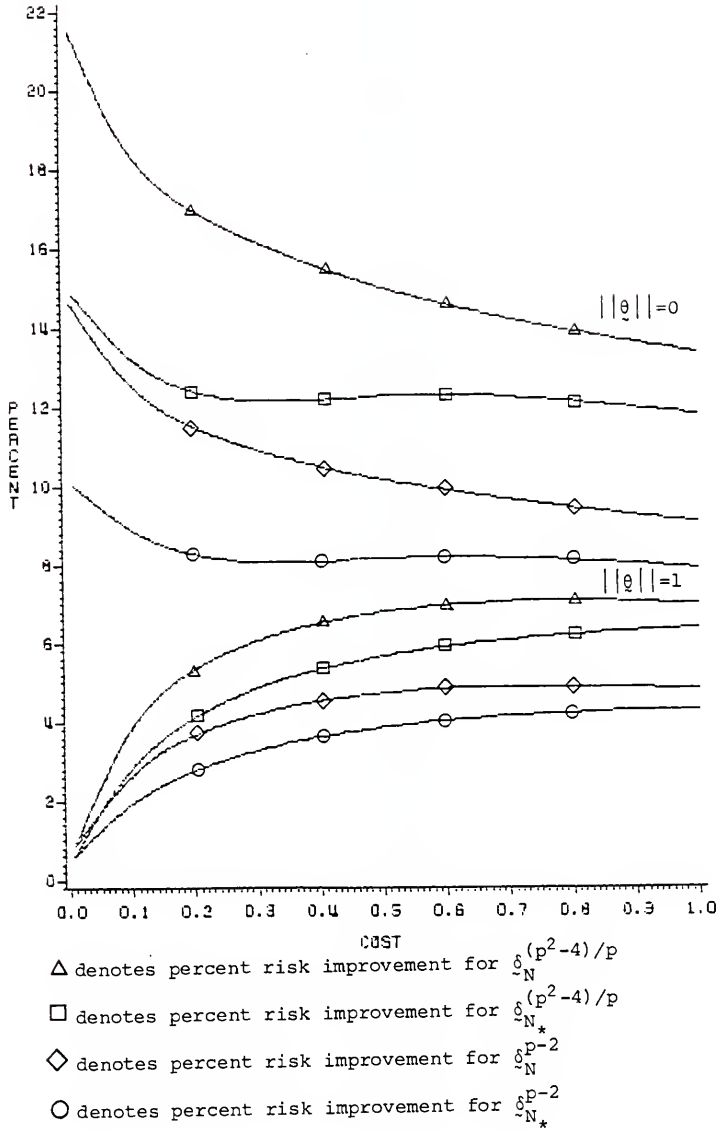


Figure 2.6 Percent Risk Improvement for All Estimators.

CHAPTER THREE
SEQUENTIAL SHRINKAGE ESTIMATION OF LINEAR REGRESSION PARAMETERS

3.1 Introduction

Consider the linear regression model $Y_i = \underset{\sim}{z}_i' \underset{\sim}{\beta} + \varepsilon_i$, $i=1,2,\ldots,$ where the ε_i's are independent and identically distributed $N(0,\sigma^2)$, $\underset{\sim}{\beta}(p\times 1)$ is unknown, and $\underset{\sim}{z}_1$, $\underset{\sim}{z}_2$, . . . are known $p\times 1$ vectors. Let $\underset{\sim}{Z}_n' = (\underset{\sim}{z}_1,\ldots,\underset{\sim}{z}_n)$ and $\underset{\sim}{Y}_n = (Y_1,\ldots,Y_n)'$. Assume that the rank of $\underset{\sim}{Z}_n' \underset{\sim}{Z}_n$ is p. The problem is estimation of $\underset{\sim}{\beta}$. Given $\underset{\sim}{Y}_n$, let the loss incurred in estimating $\underset{\sim}{\beta}$ by $\underset{\sim}{\delta}_n(\underset{\sim}{Y}_n)$ be

$$L_n(\underset{\sim}{\delta}_n,\underset{\sim}{\beta}) = \frac{1}{n}(\underset{\sim}{\delta}_n - \underset{\sim}{\beta})'(\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)(\underset{\sim}{\delta}_n - \underset{\sim}{\beta}) + cn \ . \tag{3.1}$$

The usual estimate of $\underset{\sim}{\beta}$ is the least squares estimator $\hat{\underset{\sim}{\beta}}_n = (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1}\underset{\sim}{Z}_n' \underset{\sim}{Y}_n$, which is distributed $N(\underset{\sim}{\beta},\sigma^2(\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1})$, with risk

$$R(\underset{\sim}{\beta},\sigma^2,\hat{\underset{\sim}{\beta}}_n) = EL_n(\hat{\underset{\sim}{\beta}}_n,\underset{\sim}{\beta}) \tag{3.2}$$

$$= \frac{1}{n} E[(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta})'(\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta})] + cn$$

$$= \frac{p\sigma^2}{n} + cn \ .$$

The above risk is minimized at $n = n^* \equiv (\frac{p\sigma^2}{c})^{1/2}$. However, if σ^2 is unknown there does not exist any fixed sample size which minimizes (3.2) simultaneously for all σ^2. Again, motivated from the optimal fixed sample size n^*, we propose the following sequential procedure.

$$N = \inf \{n \geq m : n \geq (\frac{p\ MSE_n}{c})^{1/2}\} \tag{3.3}$$

where $m \geq p+1$ is the initial sample size and

$$MSE_n = \| \underset{\sim}{Y}_n - \underset{\sim}{Z}_n \hat{\underset{\sim}{\beta}}_n \|^2/(n - p) \tag{3.4}$$

for each $n \geq p+1$. The above stopping rule was first considered by Mukhopadhyay (1974) who proposed the estimator $\hat{\underset{\sim}{\beta}}_N$ for $\underset{\sim}{\beta}$ and studied asymptotic properties of N and $\hat{\underset{\sim}{\beta}}_N$.

Our next step is to show that the above procedure can be motivated from a minimax criterion. With this end, first let $\underset{\sim}{\beta}$ have a $N(\underset{\sim}{\mu},\tau^2\underset{\sim}{I}_p)$ prior. Then the posterior distribution of $\underset{\sim}{\beta}$ given $\underset{\sim}{Y}_n = \underset{\sim}{y}_n$ is

$$N(\underset{\sim}{\mu} + \tau^2\underset{\sim}{Z}_n'(\tau^2\underset{\sim}{Z}_n\underset{\sim}{Z}_n' + \sigma^2\underset{\sim}{I}_n)^{-1}(\underset{\sim}{y}_n - \underset{\sim}{Z}_n\underset{\sim}{\mu}), \tag{3.5}$$

$$\tau^2(I_{\sim p} - \tau^2 Z_{\sim n}'(\tau^2 Z_{\sim n} Z_{\sim n}' + \sigma^2 I_{\sim n})^{-1} Z_{\sim n})) \ .$$

Next we prove the following matrix lemma.

Lemma 3.1

$$Z_{\sim n}'(\tau^2 Z_{\sim n} Z_{\sim n}' + \sigma^2 I_{\sim n})^{-1} = (\tau^2 Z_{\sim n}' Z_{\sim n} + \sigma^2 I_{\sim p})^{-1} Z_{\sim n}' \ .$$

Proof: Use the identity

$$(\tau^2 Z_{\sim n}' Z_{\sim n} + \sigma^2 I_{\sim p}) Z_{\sim n}' = Z_{\sim n}'(\tau^2 Z_{\sim n} Z_{\sim n}' + \sigma^2 I_{\sim n}) \ . \tag{3.6}$$

Using Lemma 3.1 it follows that

$$\tau^2 Z_{\sim n}'(\tau^2 Z_{\sim n} Z_{\sim n}' + \sigma^2 I_{\sim n})^{-1}(Y_{\sim n} - Z_{\sim n} \mu_{\sim}) \tag{3.7}$$

$$= \tau^2(\tau^2 Z_{\sim n}' Z_{\sim n} + \sigma^2 I_{\sim p})^{-1} Z_{\sim n}'(Y_{\sim n} - Z_{\sim n} \mu_{\sim})$$

$$= \tau^2(I_{\sim p} + \frac{\sigma^2}{\tau^2}(Z_{\sim n}' Z_{\sim n})^{-1})^{-1} \tau^{-2}(Z_{\sim n}' Z_{\sim n})^{-1} Z_{\sim n}'(Y_{\sim n} - Z_{\sim n} \mu_{\sim})$$

$$= (I_{\sim p} + \frac{\sigma^2}{\tau^2}(Z_{\sim n}' Z_{\sim n})^{-1})^{-1}(\hat{\beta}_{\sim n} - \mu_{\sim}) \ .$$

Also, using Lemma 3.1,

$$\underset{\sim}{I}_p - \tau^2 \underset{\sim}{Z}_n' (\tau^2 \underset{\sim}{Z}_n \underset{\sim}{Z}_n' + \sigma^2 \underset{\sim}{I}_n)^{-1} \underset{\sim}{Z}_n \tag{3.8}$$

$$= \underset{\sim}{I}_p - (\tau^2 \underset{\sim}{Z}_n' \underset{\sim}{Z}_n + \sigma^2 \underset{\sim}{I}_p)^{-1} (\tau^2 \underset{\sim}{Z}_n' \underset{\sim}{Z}_n)$$

$$= \underset{\sim}{I}_p - (\tau^2 \underset{\sim}{Z}_n' \underset{\sim}{Z}_n + \sigma^2 \underset{\sim}{I}_p)^{-1} (\tau^2 \underset{\sim}{Z}_n' \underset{\sim}{Z}_n + \sigma^2 \underset{\sim}{I}_p - \sigma^2 \underset{\sim}{I}_p)$$

$$= \sigma^2 (\tau^2 \underset{\sim}{Z}_n' \underset{\sim}{Z}_n + \sigma^2 \underset{\sim}{I}_p)^{-1}$$

$$= \frac{\sigma^2}{\tau^2} (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n + \frac{\sigma^2}{\tau^2} \underset{\sim}{I}_p)^{-1}$$

$$= \frac{\sigma^2}{\tau^2} (\underset{\sim}{I}_p + \frac{\sigma^2}{\tau^2} (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1})^{-1} (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1} .$$

Thus from (3.5) and (3.7) - (3.8) it follows that the posterior distribution of $\underset{\sim}{\beta}$ given $\underset{\sim}{Y}_n = \underset{\sim}{y}_n$ is

$$N(\underset{\sim}{\mu} + (\underset{\sim}{I}_p + \frac{\sigma^2}{\tau^2} (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1})^{-1} (\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\mu}) , \tag{3.9}$$

$$\sigma^2 (\underset{\sim}{I}_p + \frac{\sigma^2}{\tau^2} (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1})^{-1} (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1}) .$$

The loss (3.1) being quadratic, the Bayes estimator of $\underset{\sim}{\beta}$ is

$$\underset{\sim}{\delta}_n^B = \underset{\sim}{\mu} + (\underset{\sim}{I}_p + \frac{\sigma^2}{\tau^2} (\underset{\sim}{Z}_n' \underset{\sim}{Z}_n)^{-1})^{-1} (\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\mu}) ,$$

with risk $\frac{\sigma^2}{n}\,\mathrm{tr}[(\underset{\sim}{I}_p + \frac{\sigma^2}{\tau^2}(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1})^{-1}] + cn$.

Next, consider the sequence of $\{N(\underset{\sim}{\mu}, m\underset{\sim}{I}_p),\ m \geq 1\}$ priors on $\underset{\sim}{\beta}$. Then the Bayes estimator becomes

$$\underset{\sim}{\delta}_n^B = \underset{\sim}{\mu} + (\underset{\sim}{I}_p + \frac{\sigma^2}{m}(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1})^{-1}(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\mu}) ,$$

with risk $\frac{\sigma^2}{n}\,\mathrm{tr}\,[(\underset{\sim}{I}_p + \frac{\sigma^2}{m}(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1})^{-1} + cn$. Here, the Bayes risk converges to $\frac{p\sigma^2}{n} + cn$ as $m \to \infty$, and $R(\underset{\sim}{\beta}, \sigma^2, \hat{\underset{\sim}{\beta}}_n) = \frac{p\sigma^2}{n} + cn$ for all $\underset{\sim}{\beta}$. Hence n^*, the sample size, and $\hat{\underset{\sim}{\beta}}_n$, the estimator, is a sequential minimax rule. Consequently (3.3) can be viewed as an empirical minimax stopping rule.

In Section 3.2 we show the dominance of a class of estimators over $\hat{\underset{\sim}{\beta}}_N$ for the loss (3.1) and the stopping rule (3.3). In Section 3.3 we provide an asymptotic risk expansion for $\hat{\underset{\sim}{\beta}}_N$, and an asymptotic risk expansion for the shrinkage estimators dominating $\hat{\underset{\sim}{\beta}}_N$ both up to the second order terms.

3.2 A Class of James-Stein Estimators Dominating $\hat{\underset{\sim}{\beta}}_N$

In this section we consider the class of James-Stein estimators $\underset{\sim}{\delta}_N^b(\underset{\sim}{Y}_N)$ where

$$\underset{\sim}{\delta}_n^b(\underset{\sim}{Y}_n) = \hat{\underset{\sim}{\beta}}_n - \frac{bs_n^2}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})}(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda}) \tag{3.10}$$

for every $n \geq p+1$, where $s_n^2 = \frac{p}{p+2}MSE_n$, b is a constant and $\lambda \epsilon R^p$ is the known point towards which we shrink. The main result of this section is as follows:

Theorem 3.1 Under the stopping rule (3.3) with $m \geq 2p$, and under the loss (3.1),

$$R(\underset{\sim}{\beta},\sigma^2,\underset{\sim}{\delta}_N^b) < R(\underset{\sim}{\beta},\sigma^2,\hat{\underset{\sim}{\beta}}_N)$$

for every $b \epsilon (0,2(p-2))$.

Proof: Note that

$$R(\underset{\sim}{\beta},\sigma^2,\underset{\sim}{\delta}_N^b) - R(\underset{\sim}{\beta},\sigma^2,\hat{\underset{\sim}{\beta}}_N) \tag{3.11}$$

$$= b^2 E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{1}{N}\frac{s_N^4}{(\hat{\underset{\sim}{\beta}}_N - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)(\hat{\underset{\sim}{\beta}}_N - \underset{\sim}{\lambda})}\right]$$

$$- 2bE_{\underset{\sim}{\beta},\sigma^2}\left[\frac{1}{N}\frac{s_N^4}{(\hat{\underset{\sim}{\beta}}_N - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)(\hat{\underset{\sim}{\beta}}_N - \underset{\sim}{\lambda})}(\hat{\underset{\sim}{\beta}}_N - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)(\hat{\underset{\sim}{\beta}}_N - \underset{\sim}{\beta})\right]$$

$$= \sum_{n=m}^{\infty} E_{\underset{\sim}{\beta},\sigma^2}\Big[\Big\{\frac{1}{n}\frac{b^2 s_n^4}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})}$$

$$- 2\frac{1}{n}\frac{bs_n^2}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})}(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta})\Big\} I_{[N=n]}\Big] .$$

To proceed further we shall require the following lemma concerning the orthogonal decomposition of $SSE_n (=(n-p)MSE_n)$ in regression.

Lemma 3.2 Let $\underset{\sim}{Y}_n = \underset{\sim}{Z}_n\underset{\sim}{\beta} + \underset{\sim}{e}_n$, where $\underset{\sim}{e}_n \sim N(\underset{\sim}{0}, \sigma^2\underset{\sim}{I}_n)$ for every $n \geq p+1$. Assume that $(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)$ to be invertable for every $n \geq p+1$ and define $SSE_n = \|\underset{\sim}{Y}_n - \underset{\sim}{Z}_n\hat{\underset{\sim}{\beta}}_n\|^2$. Then the following results hold.

(i) $\underset{\sim}{Z}_{n+1}'\underset{\sim}{Z}_{n+1} = \underset{\sim}{Z}_n'\underset{\sim}{Z}_n + \underset{\sim}{z}_{n+1}\underset{\sim}{z}_{n+1}'$;

(ii) $(\underset{\sim}{Z}_{n+1}'\underset{\sim}{Z}_{n+1})^{-1} = (\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}(\underset{\sim}{I}_p - \frac{1}{\eta_n}\underset{\sim}{z}_{n+1}\underset{\sim}{z}_{n+1}'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1})$, where

$\eta_n = 1 + \underset{\sim}{z}_{n+1}'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1}$;

(iii) $\hat{\underset{\sim}{\beta}}_{n+1} = \hat{\underset{\sim}{\beta}}_n + \eta_n^{-1}\varepsilon_n(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1}$, where $\varepsilon_n = Y_{n+1} - \underset{\sim}{z}_{n+1}'\hat{\underset{\sim}{\beta}}_n$ and

$\hat{\underset{\sim}{\beta}}_n = (\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{Z}_n'\underset{\sim}{Y}_n$;

(iv) $SSE_{n+1} = SSE_n + \frac{\varepsilon_n^2}{\eta_n}$;

(v) $\underset{\sim}{Y}_n - \underset{\sim}{Z}_n\hat{\underset{\sim}{\beta}}_n$ is distributed independently of ε_n.

Proof of (i): Note that $\underset{\sim}{Z}_{n+1} = (\underset{\sim}{Z}_n'\underset{\sim}{z}_{n+1})'$. Then

$$\underset{\sim}{Z}_{n+1}'\underset{\sim}{Z}_{n+1} = (\underset{\sim}{Z}_n'\underset{\sim}{z}_{n+1})\begin{bmatrix} \underset{\sim}{Z}_n \\ \underset{\sim}{z}_{n+1}' \end{bmatrix} \tag{3.12}$$

$$= \underset{\sim}{Z}_n'\underset{\sim}{Z}_n + \underset{\sim}{z}_{n+1}\underset{\sim}{z}_{n+1}' .$$

Proof of (ii): Using the matrix identity from Rao (1965) that for any matrix $\underset{\sim}{A}$ (pxp) and (px1) vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$,

$$(\underset{\sim}{A} + \underset{\sim}{u}\underset{\sim}{v}')^{-1} = \underset{\sim}{A}^{-1} - \frac{1}{1+\underset{\sim}{v}'\underset{\sim}{A}^{-1}\underset{\sim}{u}}(\underset{\sim}{A}^{-1}\underset{\sim}{u}\underset{\sim}{v}'\underset{\sim}{A}^{-1}), \tag{3.13}$$

then

$$(\underset{\sim}{Z}_{n+1}'\underset{\sim}{Z}_{n+1})^{-1} = (\underset{\sim}{Z}_n'\underset{\sim}{Z}_n + \underset{\sim}{z}_{n+1}\underset{\sim}{z}_{n+1}')^{-1} \tag{3.14}$$

$$= (\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1} - \frac{1}{1+\underset{\sim}{z}_{n+1}'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1}}[(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1}\underset{\sim}{z}_{n+1}'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}]$$

$$= (\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}(\underset{\sim}{I}_p - \frac{1}{n_n}\underset{\sim}{z}_{n+1}\underset{\sim}{z}_{n+1}'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}) .$$

Proof of (iii): Note that

$$\hat{\underset{\sim}{\beta}}_{n+1} = (\underset{\sim}{Z}'_{n+1}\underset{\sim}{Z}_{n+1})^{-1}\underset{\sim}{Z}'_{n+1}\underset{\sim}{Y}_{n+1} \tag{3.15}$$

$$= (\underset{\sim}{Z}'_n\underset{\sim}{Z}_n)^{-1}(\underset{\sim}{I}_p - \frac{1}{\eta_n}\underset{\sim}{z}_{n+1}\underset{\sim}{z}'_{n+1}(\underset{\sim}{Z}'_n\underset{\sim}{Z}_n)^{-1})(\underset{\sim}{Z}'_n\underset{\sim}{Y}_n + Y_{n+1}\underset{\sim}{z}_{n+1})$$

$$= (\underset{\sim}{I}_p - \frac{1}{\eta_n}(\underset{\sim}{Z}'_n\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1}\underset{\sim}{z}'_{n+1})\hat{\underset{\sim}{\beta}}_n$$

$$+ Y_{n+1}(\underset{\sim}{Z}'_n\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1}(1 - \frac{1}{\eta_n}\underset{\sim}{z}'_{n+1}(\underset{\sim}{Z}'_n\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1})$$

$$= \hat{\underset{\sim}{\beta}}_n - \frac{1}{\eta_n}(\underset{\sim}{Z}'_n\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1}(\underset{\sim}{z}'_{n+1}\hat{\underset{\sim}{\beta}}_n)$$

$$+ \frac{1}{\eta_n}Y_{n+1}(\underset{\sim}{Z}'_n\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1}$$

$$= \hat{\underset{\sim}{\beta}}_n + \eta_n^{-1}\varepsilon_n(\underset{\sim}{Z}'_n\underset{\sim}{Z}_n)^{-1}\underset{\sim}{z}_{n+1} ,$$

where in the fourth equality in the rhs of (3.15) we use the definition of η_n.

Proof of (iv): Note that

$$SSE_{n+1} = \|\underset{\sim}{Y}_{n+1} - \underset{\sim}{Z}_{n+1}\hat{\underset{\sim}{\beta}}_{n+1}\|^2 \tag{3.16}$$

$$= \left\| \begin{matrix} \underset{\sim}{Y}_n - \underset{\sim}{Z}_n\hat{\underset{\sim}{\beta}}_{n+1} \\ Y_{n+1} - \underset{\sim}{z}'_{n+1}\hat{\underset{\sim}{\beta}}_{n+1} \end{matrix} \right\|^2$$

$$= \|Y_{\sim n} - Z_{\sim n}\hat{\beta}_{\sim n} - \eta_n^{-1}\varepsilon_n Z_{\sim n}(Z_{\sim n}'Z_{\sim n})^{-1}z_{\sim n+1}\|^2$$

$$+ (Y_{n+1} - z_{\sim n+1}'\hat{\beta}_{\sim n} - \frac{\varepsilon_n}{\eta_n} z_{\sim n+1}'(Z_{\sim n}'Z_{\sim n})^{-1}z_{\sim n+1})^2$$

$$= \|Y_{\sim n} - Z_{\sim n}\hat{\beta}_{\sim n}\|^2 + \frac{\varepsilon_n^2}{\eta_n^2} - 2\frac{\varepsilon_n}{\eta_n}[z_{\sim n+1}'(Z_{\sim n}'Z_{\sim n})^{-1}Z_{\sim n}'(Y_{\sim n} - Z_{\sim n}\hat{\beta}_{\sim n})]$$

$$+ (\varepsilon_n - \frac{\varepsilon_n}{\eta_n} z_{\sim n+1}'(Z_{\sim n}'Z_{\sim n})^{-1}z_{\sim n+1})^2$$

$$= SSE_n + \frac{\varepsilon_n^2}{\eta_n^2}(\eta_n - 1) + (\varepsilon_n - \frac{\varepsilon_n}{\eta_n}(\eta_n - 1))^2$$

$$= SSE_n + \frac{\varepsilon_n^2}{\eta_n},$$

where in the third equality in the rhs of (3.16) we use part (iii), while in the fifth equality we use the definitions of η_n and $\hat{\beta}_{\sim n}$.

Proof of (v): Since $Y_{\sim n} - Z_{\sim n}\hat{\beta}_{\sim n}$ and ε_n are both normally distributed then to prove their independence we need only show that their covariance is zero. With this end note that

$$\text{Cov}(Y_{\sim n} - Z_{\sim n}\hat{\beta}_{\sim n}, \varepsilon_n) \tag{3.17}$$

$$= \text{Cov}(Y_{\sim n} - Z_{\sim n}\hat{\beta}_{\sim n}, Y_{\sim n+1} - z_{\sim n+1}'\hat{\beta}_{\sim n})$$

$$= -\text{Cov}(Y_{\sim n}, z_{\sim n+1}'\hat{\beta}_{\sim n}) + \text{Cov}(Z_{\sim n}\hat{\beta}_{\sim n}, z_{\sim n+1}'\hat{\beta}_{\sim n})$$

$$= - \mathrm{Var}(\underset{\sim}{Y}_n)\underset{\sim}{Z}_n(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{Z}_{n+1} + \underset{\sim}{Z}_n \mathrm{Var}(\hat{\underset{\sim}{\beta}}_n)\underset{\sim}{Z}_{n+1}$$

$$= - \sigma^2\underset{\sim}{Z}_n(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{Z}_{n+1} + \sigma^2\underset{\sim}{Z}_n(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{Z}_{n+1}$$

$$= 0 .$$

Note that if we define $X_1 = \|\underset{\sim}{Y}_{p+1} - \underset{\sim}{Z}_{p+1}\hat{\underset{\sim}{\beta}}_{p+1}\|^2$ and $X_i = \dfrac{\varepsilon_{i+p-1}^2}{\eta_{i+p-1}}$, $i=2,\ldots,n-p$, then from Lemma 3.2 the X_i's are independent, and $X_i \sim \sigma^2\chi_1^2$, $i=1,\ldots,p$. Also, from Lemma 3.2, for each $n \geq p+1$,

$$SSE_{p+1} = \|\underset{\sim}{Y}_{p+1} - \underset{\sim}{Z}_{p+1}\hat{\underset{\sim}{\beta}}_{p+1}\|^2 = X_1 ,$$

$$SSE_{p+2} = \|\underset{\sim}{Y}_{p+1} - \underset{\sim}{Z}_{p+1}\hat{\underset{\sim}{\beta}}_{p+1}\|^2 + \frac{\varepsilon_{p+1}^2}{\eta_{p+1}} = X_1 + X_2 ,$$

$$\vdots$$

$$SSE_{n-1} = \|\underset{\sim}{Y}_{n-2} - \underset{\sim}{Z}_{n-2}\hat{\underset{\sim}{\beta}}_{n-2}\|^2 + \frac{\varepsilon_{n-2}^2}{\eta_{n-2}} = X_1+X_2+\ldots+X_{n-p-1} ,$$

and

$$SSE_n = \|\underset{\sim}{Y}_{n-1} - \underset{\sim}{Z}_{n-1}\hat{\underset{\sim}{\beta}}_{n-1}\|^2 + \frac{\varepsilon_{n-1}^2}{\eta_{n-1}} = X_1+X_2+\ldots+X_{n-p-1}+X_{n-p} .$$

Consequently, $MSE_n = (n-p)^{-1}SSE_n$ can be considered as the mean of $(n-p)$ independent and identically distributed χ_1^2 random variables.

Denote by B_n the σ-algebra generated by $X_1, \ldots, X_{n-p}$. Then using Lemma 3.2,

$$E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{s_n^4}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})} I_{[N=n]}\right] \tag{3.18}$$

$$= E_{\underset{\sim}{\beta},\sigma^2} E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{s_n^4}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})} I_{[N=n]} | B_n\right]$$

$$= E_{\underset{\sim}{\beta},\sigma^2}\left[s_n^4 I_{[N=n]} E_{\underset{\sim}{\beta},\sigma^2}\left\{\frac{1}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})} | B_n\right\}\right]$$

$$= E_{\sigma^2}\left[s_n^4 I_{[N=n]}\right] E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{1}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})}\right],$$

where for the second equality in (3.18) we use the fact that $I_{[N=n]}$ is a function of $X_1, \ldots, X_{n-p}$ only, and in the third equality we use the independence of $X_1, \ldots, X_{n-p}$ and $\hat{\underset{\sim}{\beta}}_n$.

Next, note that for every $n \geq p+1$, since $\underset{\sim}{Z}_n'\underset{\sim}{Z}_n$ is positive definite, using the diagonalization theorem, there exists a nonsingular $\underset{\sim}{D}_n$ such that $\underset{\sim}{D}_n'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)^{-1}\underset{\sim}{D}_n = \underset{\sim}{A}_n^{-1}$, a diagonal matrix with all positive diagonal elements. We write a_{ni} as the i-th diagonal element of $\underset{\sim}{A}_n$. Use the transformation $\underset{\sim}{L}_n = \underset{\sim}{D}_n'(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})$, so that $\underset{\sim}{L}_n \sim N(\underset{\sim}{\xi}_n, \sigma^2\underset{\sim}{A}_n^{-1})$, with $\underset{\sim}{\xi}_n = \underset{\sim}{D}_n'(\underset{\sim}{\beta}_n - \underset{\sim}{\lambda})$. Then

$$E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{\sigma^2}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})}\right] = E_{\underset{\sim}{\xi}_n,\sigma^2}\left[\frac{\sigma^2}{\underset{\sim}{L}_n'\underset{\sim}{A}_n\underset{\sim}{L}_n}\right]. \tag{3.19}$$

Here, for every n, $\frac{\underset{\sim}{L}_n'\underset{\sim}{A}_n\underset{\sim}{L}_n}{\sigma^2} \sim \Sigma_{i=1}^{p} W_{ni}$, where the W_{ni}'s are independent and $W_{ni} \sim \chi_1^2(\frac{\xi_{ni}^2}{2\sigma^2/a_{ni}})$, i=1,...,p. Hence, using again the result that if P(U>0) = 1 then $E(U^{-1}) = \int_0^\infty E[\exp(-tU)]dt$, one gets

$$\text{lhs of (3.19)} = E_{\underset{\sim}{\xi}_n,\sigma^2}\left[\frac{\underset{\sim}{L}_n'\underset{\sim}{A}_n\underset{\sim}{L}_n}{\sigma^2}\right]^{-1} \tag{3.20}$$

$$= \int_0^\infty E[\exp(-t\Sigma_{i=1}^{p} W_{ni})]dt$$

$$= \int_0^\infty \prod_{i=1}^{p} E[\exp(-tW_{ni})]dt$$

$$= \int_0^\infty \prod_{i=1}^{p} \{(\frac{1}{1+2t})^{1/2} \exp(-\frac{\xi_{ni}^2}{2\sigma^2/a_{ni}} \frac{2t}{1+2t})\}dt$$

$$= \int_0^\infty (\frac{1}{1+2t})^{p/2} \exp\left(-\frac{t}{1+2t}\frac{1}{\sigma^2}\Sigma_{i=1}^{p} a_{ni}\xi_{ni}^2\right)dt$$

$$= \int_0^\infty (\frac{1}{1+2t})^{p/2} \exp(-\frac{t}{1+2t}\frac{1}{\sigma^2}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\underset{\sim}{\beta}-\underset{\sim}{\lambda}))dt$$

$$= g_p(\Delta_n) \quad \text{(say)},$$

since $\Sigma_{i=1}^{p} a_{ni}\xi_{ni}^2 = \underset{\sim}{\xi}_n'\underset{\sim}{A}_n\underset{\sim}{\xi}_n = (\underset{\sim}{\beta}-\underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\underset{\sim}{\beta}-\underset{\sim}{\lambda})$. Here, $\Delta_n = \Delta_n(\underset{\sim}{\beta},\sigma^2)$.

We shall next show that $g_p(\Delta_n)$ is nonincreasing in n. Note that for any nxp design matrix $\underset{\sim}{Z}_n$,

$$\underset{\sim}{Z}_n'\underset{\sim}{Z}_n = \begin{bmatrix} n & \Sigma_{i=1}^n z_{1i} & \cdots & \Sigma_{i=1}^n z_{p-1,i} \\ \Sigma_{i=1}^n z_{1i} & \Sigma_{i=1}^n z_{1i}^2 & \cdots & \Sigma_{i=1}^n z_{1i} z_{p-1,i} \\ \vdots & \vdots & & \vdots \\ \Sigma_{i=1}^n z_{p-1,i} & \cdots & & \Sigma_{i=1}^n z_{p-1,i}^2 \end{bmatrix} . \tag{3.21}$$

Then for any $\underset{\sim}{\alpha} \in R^p$,

$$\begin{aligned} \underset{\sim}{\alpha}'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)\underset{\sim}{\alpha} &= \Sigma_{i=1}^p \Sigma_{j=1}^p \alpha_i (\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)_{ij} \alpha_j \\ &= \Sigma_{i=1}^p \alpha_i^2 (\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)_{ii} + 2 \sum_{i<j} \alpha_i \alpha_j (\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)_{ij} \\ &= \Sigma_{i=1}^p \alpha_i (\Sigma_{k=1}^n z_{ik}^2) + 2 \sum_{i<j} \alpha_i \alpha_j (\Sigma_{k=1}^n z_{ik} z_{jk}) \\ &= \Sigma_{k=1}^n [\alpha_1 + \alpha_2 z_{1k} + \cdots + \alpha_p z_{p-1,k}]^2 \end{aligned} \tag{3.22}$$

which is a nondecreasing function of n. Hence, using (3.21) and (3.22), $g_p(\Delta_n)$ is a nonincreasing function of n.

Next, observe that

$$E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{s_n^2}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})} (\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}) I_{[N=n]}\right] \tag{3.23}$$

$$= E_{\underset{\sim}{\beta},\sigma^2} E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{s_n^2}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(Z_{\sim n}'Z_{\sim n})(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})}(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(Z_{\sim n}'Z_{\sim n})(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta}) I_{[N=n]} \mid B_n\right]$$

$$= E_{\sigma^2}[s_n^2 I_{[N=n]}] E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(Z_{\sim n}'Z_{\sim n})(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta})}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(Z_{\sim n}'Z_{\sim n})(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})}\right]$$

by the independence of $X_1, \ldots, X_{n-p}$ and $\hat{\underset{\sim}{\beta}}_n$, and that $I_{[N=n]}$ is a function only of $X_1, \ldots, X_{n-p}$.

Using the diagonalization as before,

$$E_{\underset{\sim}{\beta},\sigma^2}\left[\frac{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(Z_{\sim n}'Z_{\sim n})(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\beta})}{(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})'(Z_{\sim n}'Z_{\sim n})(\hat{\underset{\sim}{\beta}}_n - \underset{\sim}{\lambda})}\right] \tag{3.24}$$

$$= E_{\underset{\sim}{\xi}_n,\sigma^2}\left[\frac{L_{\sim n}'A_{\sim n}(L_{\sim n} - \underset{\sim}{\xi}_n)}{L_{\sim n}'A_{\sim n}L_{\sim n}}\right]$$

$$= \Sigma_{i=1}^{p} E_{\underset{\sim}{\xi}_n,\sigma^2}\left[\frac{1}{L_{\sim n}'A_{\sim n}L_{\sim n}} L_{ni}(L_{ni} - \xi_{ni})a_{ni}\right]$$

$$= \Sigma_{i=1}^{p} E_{\underset{\sim}{\xi}_n,\sigma^2}\left[\sigma^2 \frac{\partial}{\partial L_{ni}} \frac{L_{ni}}{L_{\sim n}'A_{\sim n}L_{\sim n}}\right]$$

$$= \Sigma_{i=1}^{p} \sigma^2 E_{\underset{\sim}{\xi}_n,\sigma^2}\left[\frac{1}{L_{\sim n}'A_{\sim n}L_{\sim n}} - \frac{L_{ni}}{(L_{\sim n}'A_{\sim n}L_{\sim n})^2} 2L_{ni}a_{ni}\right]$$

$$= (p-2)E_{\underset{\sim}{\xi}_n,\sigma^2}\left[\frac{\sigma^2}{\underset{\sim}{L}_n'\underset{\sim}{A}_n\underset{\sim}{L}_n}\right]$$

$$= (p-2)g_p(\Delta_n) .$$

For the third equality in (3.24), one uses Stein's identity, while for the last equality, one uses (3.19) and (3.20).

Combining (3.18) - (3.20) and (3.23) - (3.24),

$$R(\underset{\sim}{\beta},\sigma^2,\underset{\sim}{\delta}^b_N) - R(\underset{\sim}{\beta},\sigma^2,\underset{\sim}{\hat{\beta}}_N) \tag{3.25}$$

$$= b\,\Sigma_{n=m}^{\infty}\, g_p(\Delta_n)E_{\sigma^2}\left[\frac{s_n^2}{n}\left\{b\,\frac{s_n^2}{\sigma^2} - 2(p-2)\right\}I_{[N=n]}\right]$$

$$< 2(p-2)b\,\Sigma_{n=m}^{\infty} g_p(\Delta_n)\,E_{\sigma^2}\left[\frac{s_n^2}{n}\left\{\frac{s_n^2}{\sigma^2} - 1\right\}\,I_{[N=n]}\right] ,$$

where in the last step one uses $0 < b < 2(p-2)$. Accordingly, for proving the theorem, it suffices to show that

$$\Sigma_{n=m}^{\infty} g_p(\Delta_n)E_{\sigma^2}\left[\frac{s_n^2}{n}\left\{\frac{s_n^2}{\sigma^2} - 1\right\}I_{[N=n]}\right] \leq 0 \text{ for all } \sigma^2 > 0 . \tag{3.26}$$

With this end, let n_o be the smallest integer $\geq m$ ($\geq 2p$) such that $p(p+2)^{-1}\, c\, n_o^2/p \geq \sigma^2$. Then, write

$$\text{lhs of (3.26)} = \sum_{n=m}^{n_o-1} g_p(\Delta_n) E_{\sigma^2}[\frac{s_n^2}{n}\{\frac{s_n^2}{\sigma^2} - 1\}I_{[N=n]}] \tag{3.27}$$

$$+ g_p(\Delta_{n_o}) E_{\sigma^2}[\frac{s_{n_o}^2}{n_o}\{\frac{s_{n_o}^2}{\sigma^2} - 1\}I_{[N\geq n_o]}]$$

$$+ \sum_{n=n_o}^{\infty}\{g_p(\Delta_{n+1}) E_{\sigma^2}[\frac{s_{n+1}^2}{n+1}\{\frac{s_{n+1}^2}{\sigma^2} - 1\}I_{[N\geq n+1]}]$$

$$- g_p(\Delta_n) E_{\sigma^2}[\frac{s_n^2}{n}\{\frac{s_n^2}{\sigma^2} - 1\}I_{[N\geq n+1]}] ,$$

where the first term in the rhs of (3.27) should be interpreted as zero if $n_o = m$. Note that for $n \geq n_o$, on the set $[N \geq n+1]$, $s_n^2 \geq p(p+2)^{-1}cn^2/p \geq p(p+2)^{-1}cn_o^2/p \geq \sigma^2$. Also, by (3.21) and (3.22), $g_p(\Delta_n)$ is nonincreasing in n. Hence,

$$\text{third term in the rhs of (3.27)} \tag{3.28}$$

$$\leq \sum_{n=n_o}^{\infty} g_p(\Delta_{n+1}) E_{\sigma^2}\{[\frac{s_{n+1}^2}{n+1}\{\frac{s_{n+1}^2}{\sigma^2} - 1\} - \frac{s_n^2}{n}\{\frac{s_n^2}{\sigma^2} - 1\}]I_{[N\geq n+1]}\} .$$

Note that $I_{[N\geq n+1]}$ is a B_n measurable function. Then using the representation $s_{n+1}^2 = \frac{1}{n-p+1}\{(n-p)s_n^2 + \frac{p}{p+2}X_{n-p+1}\}$, one gets with probability 1,

$$E_{\sigma^2}[\frac{s_{n+1}^2}{n+1}\{\frac{s_{n+1}^2}{\sigma^2} - 1\}I_{[N\geq n+1]}|B_n] \tag{3.29}$$

$$= I_{[N\geq n+1]}\Big[\frac{(n-p)^2 s_n^4 + 2(n-p)s_n^2 \frac{p}{p+2}\sigma^2 + 3\frac{p^2}{(p+2)^2}\sigma^4}{(n+1)(n-p+1)^2\sigma^2}$$

$$- \frac{(n-p)s_n^2 + \frac{p}{p+2}\sigma^2}{(n+1)(n-p+1)}\Big]$$

$$= I_{[N\geq n+1]}\frac{s_n^2}{n}\Big\{\frac{s_n^2}{\sigma^2} - 1\Big\} + I_{[N\geq n+1]}\Big[\Big(\frac{1}{n+1}\frac{(n-p)^2}{(n+1-p)^2} - \frac{1}{n}\Big)\frac{s_n^4}{\sigma^2}$$

$$- \Big(\frac{1}{n+1}\frac{(n-p)((n-p+1)-2\frac{p}{p+2})}{(n-p+1)^2} - \frac{1}{n}\Big)s_n^2$$

$$- \frac{1}{n+1}\frac{((n+1-p)-3\frac{p}{p+2})}{(n+1-p)^2}\frac{p}{p+2}\sigma^2\Big] .$$

Note that the multiple of $I_{[N\geq n+1]}$ in the extreme rhs of (3.29) is a concave function of s_n^2, where the maximum occurs at

$$s_n^2 = \frac{(n+1)(n+1-p)^2 - n(n-p)((n+1-p)-2\frac{p}{p+2})}{2((n+1)(n+1-p)^2 - n(n-p)^2)}\sigma^2 < \sigma^2 . \qquad (3.30)$$

Here, recalling that on the set $[N\geq n+1]$, $s_n^2 \geq \sigma^2$, it follows that

2nd term of the extreme rhs of (3.29) (3.31)

$$\leq I_{[N\geq n+1]}\Big[\Big(\frac{1}{n+1}\frac{(n-p)^2}{(n+1-p)^2} - \frac{1}{n}\Big)\sigma^2 - \Big(\frac{1}{n+1}\frac{(n-p)((n+1-p)-2\frac{p}{p+2})}{(n+1-p)^2} - \frac{1}{n}\Big)\sigma^2$$

$$- \frac{1}{n+1} \frac{((n+1-p)-3\frac{p}{p+2})}{(n+1-p)^2} \frac{p}{p+2} \sigma^2]$$

$$= I_{[N \geq n+1]} (\frac{1}{n+1} \frac{1}{(n+1-p)^2})[3\frac{p}{p+2} - \frac{2}{p}(n-p)-1]\frac{p}{p+2} \sigma^2$$

$$\leq 0 ,$$

since $n \geq 2p$. Hence, from (3.28), (3.29) and (3.31),

$$\text{third term in the rhs of (3.27)} \leq 0 . \tag{3.32}$$

Next, note that $I_{[N \geq m]} = 1$ with probability 1 and

$$E_{\sigma^2}[\frac{s_m^2}{m}(\frac{s_m^2}{\sigma^2} - 1)] = \frac{\sigma^2}{m}[(\frac{p}{p+2})^2 \frac{1}{m-p}(m-p+2) - \frac{p}{p+2}] \tag{3.33}$$

$$= \frac{\sigma^2}{m}(\frac{p}{p+2})^2 \frac{1}{m-p}[2 - \frac{2}{p}(m-p)]$$

$$\leq 0 ,$$

since $m \geq 2p$. Thus, if $n_o = m$ then rhs of (3.27) ≤ 0. For $n_o > m$, first note that for $n \leq n_o - 1$, on the set $[N=n]$, $s_n^2 \leq p(p+2)^{-1}cn^2/p \leq p(p+2)^{-1}c(n_o-1)^2/p < \sigma^2$ so that using the fact that $g_p(\Delta_n)$ is nonincreasing in n,

$$\text{the first term in the rhs of (3.27)} \tag{3.34}$$

$$\leq g_p(\Delta_n) \sum_{n=m}^{n_o-1} E_{\sigma^2}[\frac{s_n^2}{n}(\frac{s_n^2}{\sigma^2} - 1)I_{[N=n]}] .$$

In view of (3.26), (3.27), (3.32) and (3.34), for proving the theorem it suffices to show that

$$\Sigma_{n=m}^{n_o-1} E_{\sigma^2}[\frac{s_n^2}{n}(\frac{s_n^2}{\sigma^2} - 1)I_{[N=n]}] + E_{\sigma^2}[\frac{s_{n_o}^2}{n_o}(\frac{s_{n_o}^2}{\sigma^2} - 1)I_{[N\geq n_o]}] \leq 0 . \quad (3.35)$$

To prove (3.35), first writing $s_{n_o}^2 = \frac{1}{n_o-p}[(n_o-1-p)s_{n_o-1}^2 + \frac{p}{p+2} X_{n_o-p}]$, one gets

$$E_{\sigma^2}[\frac{s_{n_o}^2}{n_o}(\frac{s_{n_o}^2}{\sigma^2} - 1)I_{[N\geq n_o]}] \quad (3.36)$$

$$= E_{\sigma^2}E_{\sigma^2}[\frac{s_{n_o}^2}{n_o}(\frac{s_{n_o}^2}{\sigma^2} - 1)I_{[N\geq n_o]}|B_{n_o-1}]$$

$$= E_{\sigma^2}[\{\frac{(n_o-1-p)^2 s_{n_o-1}^4 + 2(n_o-1-p)s_{n_o-1}^2 \frac{p}{p+2}\sigma^2 + 3(\frac{p}{p+2})^2\sigma^4}{n_o(n_o-p)^2\sigma^2}$$

$$- \frac{(n_o-1-p)s_{n_o-1}^2 + \frac{p}{p+2}\sigma^2}{n_o(n_o-p)}\}I_{[N\geq n_o]}]$$

$$= E_{\sigma^2}[\{b_{n_o}\frac{s_{n_o-1}^4}{\sigma^2} - c_{n_o}s_{n_o-1}^2 - d_{n_o}\sigma^2\}I_{[N\geq n_o]}] ,$$

where $b_{n_o} = \frac{1}{n_o} \frac{(n_o-1-p)^2}{(n_o-p)^2}$, $c_{n_o} = \frac{1}{n_o} \frac{(n_o-1-p)(n_o-p-2\frac{p}{p+2})}{(n_o-p)^2}$, and

$d_{n_o} = \frac{1}{n_o} \frac{(n_o-p-3\frac{p}{p+2})}{(n_o-p)^2} \frac{p}{p+2}$. Note that

$$c_{n_o} - b_{n_o} = \frac{n_o-1-p}{n_o(n_o-p)^2}[1 - 2\frac{p}{p+2}] < 0 \tag{3.37}$$

and that

$$d_{n_o} + (c_{n_o} - b_{n_o}) = \frac{1}{n_o(n_o-p)^2}[2(p-1)(n_o-p)-3\frac{p^2}{p+2} -(p-2)] > 0 \,, \tag{3.38}$$

since $n_o \geq 2p$. Also, let

$$f_{n_o} = (n_o-1)(2b_{n_o} - c_{n_o}) = \frac{n_o-1}{n_o} \frac{(n_o-1-p)}{(n_o-p)} \frac{(n_o-p-\frac{4}{p})}{(n_o-p)} \,. \tag{3.39}$$

So that $f_{n_o} \in (0,1)$. Now, rewrite

extreme right of (3.36) (3.40)

$$= E_{\sigma^2}[\{(n_o-1)(2b_{n_o} - c_{n_o})\frac{s^2_{n_o-1}}{n_o-1}(\frac{s^2_{n_o-1}}{\sigma^2} - 1)$$

$$+(c_{n_o}-b_{n_o})\{\frac{s^4_{n_o-1}}{\sigma^2} - 2s^2_{n_o-1} + \sigma^2\} - (d_{n_o}+c_{n_o}-b_{n_o})\sigma^2\}I_{[N\geq n_o]}]$$

$$\leq f_{n_o} E_{\sigma^2}[\frac{s^2_{n_o-1}}{n_o-1}(\frac{s^2_{n_o-1}}{\sigma^2} - 1)I_{[N\geq n_o]}] \ .$$

If $E_{\sigma^2}[\frac{s^2_{n_o-1}}{n_o-1}(\frac{s^2_{n_o-1}}{\sigma^2} - 1)I_{[N\geq n_o]}] \leq 0$, noting again that $s^2_n < \sigma^2$ on the set $[N=n]$ for all $n \leq n_o-1$, one proves (3.35) from (3.36) and (3.40). Otherwise using $f_{n_o} < 1$, one gets from (3.35), (3.36) and (3.40) that

$$\text{lhs of (3.35)} \tag{3.41}$$

$$\leq \Sigma_{n=m}^{n_o-2} E_{\sigma^2}[\frac{s^2_n}{n}(\frac{s^2_n}{\sigma^2} - 1)I_{[N=n]}] + E_{\sigma^2}[\frac{s^2_{n_o-1}}{n_o-1}(\frac{s^2_{n_o-1}}{\sigma^2} - 1)I_{[N\geq n_o-1]}] \ .$$

Proceed inductively to get either lhs of (3.36) ≤ 0, or finally end with

$$\text{lhs of (3.35)} \leq E_{\sigma^2}[\frac{s^2_m}{m}(\frac{s^2_m}{\sigma^2} - 1)I_{[N\geq m]}] \leq 0 \tag{3.42}$$

as shown earlier.

Remark. Initially we defined $S_n^2 = \frac{1}{(n-p)+a_n} MSE_n$, realizing the need for a biased estimator of σ^2 in $\delta_{\underset{\sim}{n}}^b$ from Chapter 2. The exact dominance was then proven with conditions on a_n established along the way. The particular choice of $a_n = \frac{2}{p}(n-p)$ was most appealing since then $S_n^2 = \frac{1}{(n-p)+\frac{2}{p}(n-p)} MSE_n = \frac{p}{p+2}\frac{1}{n-p} MSE_n$ and hence $E_{\sigma^2}[S_n^2] = \frac{p}{p+2}\sigma^2$. Therefore, the bias would be negligible for large p (as in Chapter 2). As a consequence of our choice we needed $m \geq 2p$, and thus we require such a large initial sample size.

3.3 Asymptotic Risk Expansion for $\hat{\underset{\sim}{\beta}}_N$ and $\delta_{\underset{\sim}{N}}^b$

In this section we first obtain an asymptotic risk expansion for $\hat{\underset{\sim}{\beta}}_N$ up to the second order term. Subsequently, we find an asymptotic risk expansion up to the second order term for $\delta_{\underset{\sim}{N}}^b$.

Observe that

$$R(\underset{\sim}{\beta},\sigma^2,\hat{\underset{\sim}{\beta}}_N) = p\sigma^2 E[N^{-1}] + cE[N] \tag{3.43}$$

$$= 2cn^* + cE[\frac{(N-n^*)^2}{N}]$$

$$= 2c^{1/2}(p\sigma^2)^{1/2} + cE[\frac{(N-n^*)^2}{N}] .$$

From definition $N \overset{a.s.}{\to} \infty$ as $c \to 0$ so that $MSE_N \overset{a.s.}{\to} \sigma^2$ as $c \to 0$, since MSE_n is the mean of n-p independent and identically distributed

random variables. Also, using Amscombe's Theorem, $\sqrt{N-p}\ (MSE_N - \sigma^2) \xrightarrow{L}$ $N(0,2\sigma^4)$ so that using the delta method, $\sqrt{N-p}\ (MSE_N^{1/2} - \sigma) \xrightarrow{L}$ $N(0, 1/2\ \sigma^2)$.

Next use the inequality

$$(p/c)^{1/2} MSE_N^{1/2} \leq N \leq m + (p/c)^{1/2} MSE_{N-1}^{1/2} . \tag{3.44}$$

Dividing both sides of (3.44) by n*, making $c \to 0$, and using $MSE_N^{1/2} \xrightarrow{a.s.} \sigma$ as $c \to 0$, it follows that $N/n^* \xrightarrow{a.s.} 1$ as $c \to 0$. Thus, $\sqrt{n^*-p}\ (MSE_N^{1/2} - \sigma) \xrightarrow{L} N(0, 1/2\ \sigma^2)$, and also $\sqrt{n^*-p}\ (MSE_{N-1}^{1/2} - \sigma) \xrightarrow{L}$ $N(0, 1/2\ \sigma^2)$.

Again from (3.44), one gets the inequality

$$n^{*1/2}\ \frac{MSE_N^{1/2} - \sigma}{\sigma} \leq \frac{N-n^*}{n^{*1/2}} \leq \frac{m}{n^{*1/2}} + n^{*1/2}\ \frac{MSE_{N-1}^{1/2} - \sigma}{\sigma} , \tag{3.45}$$

implying that $(N-n^*)/n^{*1/2} \xrightarrow{L} N(0, 1/2)$ as $c \to 0$. Hence $(N-n^*)/N^{1/2} \xrightarrow{L}$ $N(0, 1/2)$ since $(N/n^*) \xrightarrow{a.s.} 1$ as $c \to 0$. Also, the uniform integrability of $\frac{(N-n^*)^2}{N}$ for $c \leq c_o$ (specified) can be proved by repeating essentially the arguments of Ghosh and Mukhopadhyay (1980). Thus, $E_{\sigma^2}[\frac{(N-n^*)^2}{N}] \to 1/2$ as $c \to 0$, and it follows that from (3.44) as $c \to 0$,

$$R(\underset{\sim}{\beta},\sigma^2,\hat{\underset{\sim}{\beta}}_N) = 2c^{1/2}(p\sigma^2)^{1/2} + 1/2\, c + o(c) . \tag{3.46}$$

Next, we find the asymptotic risk expansion for $\delta^b_{\underset{\sim}{N}}$. It follows from (3.11), (3.18) - (3.20) and (3.23) - (3.25) that

$$R(\underset{\sim}{\beta},\sigma^2,\delta^b_{\underset{\sim}{N}}) = R(\underset{\sim}{\beta},\sigma^2,\hat{\underset{\sim}{\beta}}_N) \tag{3.47}$$

$$- 2b(p-2)E_{\sigma^2}[\frac{s_N^2}{N}\int_0^\infty (\frac{1}{1+2t})^{p/2}\exp\{-\frac{N}{\sigma^2}\frac{1}{1+2t}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(Z'_{\underset{\sim}{N}}Z_{\underset{\sim}{N}})\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}dt]$$

$$+ \frac{b^2}{\sigma^2}E_{\sigma^2}[\frac{s_N^4}{N}\int_0^\infty (\frac{1}{1+2t})^{p/2}\exp\{-\frac{N}{\sigma^2}\frac{t}{1+2t}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(Z'_{\underset{\sim}{N}}Z_{\underset{\sim}{N}})\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}dt] .$$

Now, for technical reasons, we shall handle the risk expansions separately for the cases $\underset{\sim}{\beta} \neq \underset{\sim}{\lambda}$ and $\underset{\sim}{\beta} = \underset{\sim}{\lambda}$. However, both cases will require the facts that as $c \to 0$, $s_N^4 \overset{a.s.}{\to} \sigma^4(\frac{p}{p+2})^2$, $s_N^2 \overset{a.s.}{\to} \sigma^2(\frac{p}{p+2})$ and $Nc^{1/2} \overset{a.s.}{\to} (p\sigma^2)^{1/2}$. Also, at this time, we shall make the following assumption that $\frac{1}{n}(Z'_{\underset{\sim}{n}}Z_{\underset{\sim}{n}}) \to \underset{\sim}{K}$ as $n \to \infty$, where $\underset{\sim}{K}$ is a positive definite matrix. Hence $\frac{1}{N}(Z'_{\underset{\sim}{N}}Z_{\underset{\sim}{N}}) \overset{a.s.}{\to} \underset{\sim}{K}$ as $c \to 0$.

Consider first the case $\underset{\sim}{\beta} \neq \underset{\sim}{\lambda}$. Here, as $c \to 0$,

$$c^{-1}\frac{s_N^4}{N}\int_0^\infty (\frac{1}{1+2t})^{p/2}\exp\{-\frac{N}{\sigma^2}\frac{t}{1+2t}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(Z'_{\underset{\sim}{N}}Z_{\underset{\sim}{N}})\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}dt \tag{3.48}$$

$$= \frac{s_N^4}{Nc^{1/2}} \int_0^\infty \left(\frac{1}{1+2tc^{1/2}}\right)^{p/2} \exp\left\{-\frac{Nc^{1/2}}{\sigma^2} \frac{t}{1+2tc^{1/2}} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})' \left\{\frac{1}{N}(Z_{\sim N}' Z_{\sim N})\right\} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})\right\} dt$$

$$\overset{a.s.}{\to} \frac{\sigma^4}{(p\sigma^2)^{1/2}} \left(\frac{p}{p+2}\right)^2 \int_0^\infty \exp\left\{-t \frac{(p\sigma^2)^{1/2}}{\sigma^2} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})' \underset{\sim}{K} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})\right\} dt$$

$$= \sigma^4 \frac{p}{(p+2)^2} \frac{1}{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})' \underset{\sim}{K} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})} .$$

Next we prove the uniform integrability (in $c \le c_0$) of the left hand side of (3.48) when $\underset{\sim}{\beta} \neq \underset{\sim}{\lambda}$. First note that

$$\text{lhs of (3.48)} \tag{3.49}$$

$$\le \frac{s_N^4}{Nc} \left[\int_0^1 \exp\left\{-\frac{Nt}{3\sigma^2} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})' \left\{\frac{1}{N}(Z_{\sim N}' Z_{\sim N})\right\} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})\right\} dt \right.$$

$$\left. + \exp\left\{-\frac{N}{3\sigma^2} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})' \left\{\frac{1}{N}(Z_{\sim N}' Z_{\sim N})\right\} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})\right\} \int_1^\infty \left(\frac{1}{1+2t}\right)^{p/2} dt\right]$$

$$\le \frac{s_N^4}{N^2 c} \frac{K_2}{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})' \left\{\frac{1}{N}(Z_{\sim N}' Z_{\sim N})\right\} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})} ,$$

where K_2 is a constant not depending on c. Thus it suffices to prove the uniform integrability of $\frac{s_N^4}{N^2 c}[(\underset{\sim}{\beta}-\underset{\sim}{\lambda})' \left\{\frac{1}{N}(Z_{\sim N}' Z_{\sim N})\right\} (\underset{\sim}{\beta}-\underset{\sim}{\lambda})]^{-1}$ in $c \le c_0$. First use the inequality

$$E_{\sigma^2}[\frac{s_N^4}{N^2c}\{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}^{-1}I_{[\frac{s_N^4}{N^2c}\{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}^{-1} > d]} \tag{3.50}$$

$$\leq d^{-\delta'} E_{\sigma^2}[(\frac{s_N^4}{N^2c}\{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}^{-1})^{1+\delta'}]$$

for some $\delta' \in (0, 1/2)$. Taking $\delta \in (\delta', 1/2)$ and using Holder's inequality, one gets

$$E_{\sigma^2}[(\frac{s_N^4}{N^2c}\{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}^{-1})^{1+\delta'}] \tag{3.51}$$

$$\leq [E_{\sigma^2}(N^2c)^{-(1+\delta)}]^{(1+\delta')/(1+\delta)}$$

$$\times [E_{\sigma^2}(s_N^4\{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}^{-1})^{\frac{(1+\delta)(1+\delta')}{\delta-\delta'}}]^{(\delta-\delta')/(1+\delta)} .$$

Taking $\delta^0 \in (\delta, 1/2)$ and using Holder's inequality once more one gets

$$\text{rhs of (3.51)} \tag{3.52}$$

$$\leq [E_{\sigma^2}(N^2c)^{-(1+\delta)}]^{(1+\delta')/(1+\delta)}$$

$$\times \{[E_{\sigma^2}(s_N^4)^{\frac{(1+\delta)(1+\delta')}{\delta-\delta'}\frac{1+\delta^0}{\delta^0-\delta}}]^{\frac{\delta^0-\delta}{1+\delta^0}}$$

$$\times \left[E_{\sigma^2}\{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}^{-\frac{(1+\delta')(1+\delta^0)}{\delta-\delta'}}\right]^{\frac{1+\delta}{1+\delta^0}\frac{\delta-\delta'}{1+\delta}} .$$

Next observe that for any arbitrary ε in $(0,1)$,

$$E_{\sigma^2}[(N^2c)^{-(1+\delta)}] \le E_{\sigma^2}[(N^2c)^{-(1+\delta)}I_{[N\le\varepsilon n^*]} + (\varepsilon^2 n^{*2}c)^{-(1+\delta)}] \quad (3.53)$$

$$\le (m^2c)^{-(1+\delta)}\, P(N\le\varepsilon n^*) + (\varepsilon^2 p\sigma^2)^{-(1+\delta)} .$$

To proceed further we need the following lemma.

Lemma 3.3 For any $0 < \varepsilon < 1$, $p \ge 1$ and $m \ge p+1$, $P(N\le\varepsilon n^*) = O(c^{(\frac{1}{2})(m-p)})$.

Proof: Note that $P_{\sigma^2}(N\le\varepsilon n^*) = \Sigma_{n=m}^{[\varepsilon n^*]} P_{\sigma^2}(N=n)$ where $[u]$ is the largest integer $\le u$. Now

$$P_{\sigma^2}(N=m) = P_{\sigma^2}(MSE_m < \frac{c}{p}m^2) \quad (3.54)$$

$$= P(\chi^2_{m-p} \le \frac{m-p}{p}\frac{c}{\sigma^2}m^2)$$

$$= O_e(c^{(\frac{1}{2})(m-p)}) ,$$

since $P(\chi^2_k \le d) = O_e(d^{k/2})$. Therefore, all we need show is that

$$\Sigma_{n=m+1}^{[\varepsilon n^*]} P_{\sigma^2}(N=n) = O(c^{(\frac{1}{2})(m-p)}) \quad . \tag{3.55}$$

Now, for $n \geq m+1$,

$$P_{\sigma^2}(N=n) = P_{\sigma^2}(MSE_n \leq \frac{c}{p} n^2) \tag{3.56}$$

$$= P(\chi^2_{n-p} \leq \frac{n-p}{p} \frac{c}{\sigma^2} n^2)$$

$$\leq \inf_{h>0} \exp[h \frac{n-p}{p} \frac{c}{\sigma^2} n^2] E[\exp(-h \chi^2_{n-p})]$$

$$= \inf_{h>0} \exp[h \frac{n-p}{p} \frac{c}{\sigma^2} n^2] (\frac{1}{1+2h})^{\frac{n-p}{2}}$$

$$= \exp[\frac{n-p}{2}(1 - \frac{n^2}{p} \frac{c}{\sigma^2})] (\frac{n^2}{p} \frac{c}{\sigma^2})^{\frac{n-p}{2}}$$

$$= [\exp(1 - \frac{n^2}{p} \frac{c}{\sigma^2})(\frac{n^2}{p} \frac{c}{\sigma^2})]^{\frac{n-p}{2}} \quad .$$

Note that for $n \leq [\varepsilon n^*]$, $\frac{n^2}{p} \frac{c}{\sigma^2} \leq \varepsilon^2 < 1$. Also, $x \exp(1-x)$ is increasing in x for $0 < x \leq 1$. Therefore

$$\Sigma_{n=m+1}^{[\varepsilon n^*]} P_{\sigma^2}(N=n) \leq \Sigma_{n=m+1}^{[\varepsilon n^*]} [\exp(1 - \frac{n^2}{p} \frac{c}{\sigma^2})(\frac{n^2}{p} \frac{c}{\sigma^2})]^{\frac{n-p}{2}} \tag{3.57}$$

$$\leq c^{\frac{m-p}{2}} (e \frac{1}{p} \frac{1}{\sigma^2})^{\frac{m-p}{2}} \Sigma_{n=m+1}^{[\varepsilon n^*]} n^{m-p} [\exp(1 - \varepsilon^2)\varepsilon^2]^{\frac{n-m}{2}}$$

$$\leq c^{\frac{m-p}{2}} \left(e \frac{1}{p\sigma^2}\right)^{\frac{m-p}{2}} \Sigma_{n=m+1}^{\infty} n^{m-p}[\exp(1 - \varepsilon^2)\varepsilon^2]^{\frac{n-m}{2}}$$

$$= O(c^{(1/2)(m-p)}) ,$$

since $\Sigma_{n=m+1}^{\infty} n^{m-p}[\exp(1 - \varepsilon^2)\varepsilon^2]^{\frac{n-m}{2}}$ is a convergent series. Hence, using Lemma 3.3 for $m \geq p+3$, and $c \leq c_o < 1$,

$$\text{lhs of (3.53)} \leq K_o(m^2 c_o)^{-(1+\delta)} c_o^{(1/2)(m-p)} + (\varepsilon^2 p\sigma^2)^{-(1+\delta)} \tag{3.58}$$

$$\leq K (m_o^2)^{-(1+\delta)} c_o^{1/2-\delta} + (\varepsilon^2 p\sigma^2)^{-(1+\delta)} ,$$

where $K_o (> 0)$ is a constant not depending on c. Further, since MSE_n can be expressed as a mean of independent and identically distributed χ_1^2 random variables then MSE_n is a one sample U-statistic with kernel of degree one and hence is a backward martingale. Consequently, using Doob's maximal inequality for submartingales, it follows that for every $r \geq 1$,

$$E_{\sigma^2}[s_N^{2r}] = E_{\sigma^2}[\frac{p}{p+2} MSE_N]^r \leq E_{\sigma^2}[MSE_N]^r \tag{3.59}$$

$$\leq E_{\sigma^2}[\sup_{n\geq m} MSE_n]^r$$

$$\leq K_1 E_{\sigma^2}[MSE_m]^r \ ,$$

where K_1 is a constant which only depends on r. Next observe that if $a_n \to a$ then for all $\varepsilon > 0$ there exists an m_o such that for all $n \geq m_o$, $|a_n - a| < \varepsilon$ implying that for all $n \geq m_o$, $\frac{1}{a_n} < \frac{1}{a-\varepsilon}$. Let $u = \frac{(1+\delta')(1+\delta^o)}{\delta-\delta'}$, $a_n = (\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{n}(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})$, $a = (\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\underset{\sim}{K}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})$ and $\varepsilon > 0$. Then

$$E_{\sigma^2}[\{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\tfrac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}^{-\frac{(1+\delta')(1+\delta^o)}{\delta-\delta'}}] \tag{3.60}$$

$$= E_{\sigma^2}[(\frac{1}{a_N})^u]$$

$$= E_{\sigma^2}[(\frac{1}{a_N} I_{[N<m_o]} + \frac{1}{a_N} I_{[N>m_o]})^u]$$

$$\leq 2^{u-1}\{E_{\sigma^2}[(\frac{1}{a_N} I_{[N<m_o]})^u] + E_{\sigma^2}[(\frac{1}{a_N} I_{[N>m_o]})^u]\}$$

$$\leq 2^{u-1}[(\sup_{n<m_o} \frac{1}{a_n})^u + (\frac{1}{a-\varepsilon})^u] \ ,$$

independent of c. Combining (3.50) - (3.60), the uniform integrability in $c \leq c_o$ of $\frac{s_N^4}{N^2 c}\{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\}^{-1}$ follows.

Thus, for $m \geq p+3$, $\underset{\sim}{\beta} \neq \underset{\sim}{\lambda}$, as $c \to 0$,

$$c^{-1}E_{\sigma^2}\left[\frac{S_N^4}{N}\int_0^\infty \left(\frac{1}{1+2t}\right)^{p/2} \exp\left\{-\frac{N}{\sigma^2}\frac{t}{1+2t}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\left\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\right\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\right\}dt\right] \tag{3.61}$$

$$\to \sigma^4 \frac{p}{(p+2)^2}\frac{1}{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\underset{\sim}{K}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})} .$$

Similarly, it can be shown that, as $c \to 0$,

$$c^{-1}E_{\sigma^2}\left[\frac{S_N^4}{N}\int_0^\infty \left(\frac{1}{1+2t}\right)^{p/2} \exp\left\{-\frac{N}{\sigma^2}\frac{t}{1+2t}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\left\{\frac{1}{N}(\underset{\sim}{Z}_N'\underset{\sim}{Z}_N)\right\}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})\right\}dt\right] \tag{3.62}$$

$$\to \sigma^2 \frac{1}{p+2}\frac{1}{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\underset{\sim}{K}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})} .$$

Hence, from (3.46) - (3.47) and (3.61) - (3.62) it follows that for $m \geq p+3$, $\underset{\sim}{\beta} \neq \underset{\sim}{\lambda}$, as $c \to 0$,

$$R(\underset{\sim}{\beta},\sigma^2,\underset{\sim}{\delta}_N^b) \tag{3.63}$$

$$= 2c^{1/2}(p\sigma^2)^{1/2} + 1/2\,c + cb\sigma^2 \frac{p}{(p+2)^2}\frac{1}{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\underset{\sim}{K}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})}\left(b-2\,\frac{p^2-4}{p}\right) + o(c) .$$

Consider now the case $\underset{\sim}{\beta} = \underset{\sim}{\lambda}$. It follows from (3.47) that

$$R(\underset{\sim}{\beta},\sigma^2,\underset{\sim}{\delta}^b_N) = R(\underset{\sim}{\beta},\sigma^2,\hat{\underset{\sim}{\beta}}_N) - 2b(p-2)E_{\sigma^2}[\frac{s_N^2}{N}\int_0^\infty (\frac{1}{1+2t})^{p/2}\,dt] \qquad (3.64)$$

$$+ \frac{b^2}{\sigma^2} E_{\sigma^2}[\frac{s_N^4}{N}\int_0^\infty (\frac{1}{1+2t})^{p/2}\,dt] .$$

Note that $\int_0^\infty (\frac{1}{1+2t})^{p/2}\,dt = \frac{1}{p-2}$. Hence, using the uniform integrability of $\frac{s_N^2}{Nc^{1/2}}$ and $\frac{s_N^4}{Nc^{1/2}}$ in $c \le c_o$ for $m \ge p+1$, one gets from (3.46) and (3.64) that as $c \to 0$,

$$R(\underset{\sim}{\beta},\sigma^2,\underset{\sim}{\delta}^b_N) = 2c^{1/2}(p\sigma^2)^{1/2} \qquad (3.65)$$

$$+ c^{1/2}\frac{p^{3/2}}{(p+2)^2}\sigma\frac{1}{p-2}\,b(b-2\,\frac{p^2-4}{p}) + o(c^{1/2}) .$$

From (3.63) it can be seen that, asymptotically (as $c \to 0$) the percent risk improvement of $\underset{\sim}{\delta}^b_N$ over $\hat{\underset{\sim}{\beta}}_N$ for $\underset{\sim}{\beta} \neq \underset{\sim}{\lambda}$ is

$$100\{\tfrac{1}{2}b\,\frac{p^{1/2}}{(p+2)^2}\,\frac{\sigma}{(\underset{\sim}{\beta}-\underset{\sim}{\lambda})'\underset{\sim}{K}(\underset{\sim}{\beta}-\underset{\sim}{\lambda})}\,(b - 2\,\frac{p^2-4}{p})\}c^{1/2} + o(c^{1/2}) , \qquad (3.66)$$

while from (3.65), asymptotically the percent risk improvement of $\underset{\sim}{\delta}^b_N$ over $\hat{\underset{\sim}{\beta}}_N$ for $\underset{\sim}{\beta} = \underset{\sim}{\lambda}$ is

$$100\{\tfrac{1}{2}b\,\frac{p}{(p+2)^2}\,\frac{1}{p-2}(b - 2\,\frac{p^2-4}{p})\} + o(1). \qquad (3.67)$$

Here the dominant term in both (3.66) and (3.67) is maximized when $b = (p^2-4)/p$. Therefore, from an asymptotic point of view as evident in (3.66) and (3.67) it appears that for small c, $(p^2-4)/p$ is an optimal choice for b (as in Chapter 2).

CHAPTER FOUR
SEQUENTIAL SHRINKAGE ESTIMATION OF THE DIFFERENCE OF TWO MULTIVARIATE NORMAL MEANS

4.1 Introduction

Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ and $\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \ldots$ be two independent sequences of random vectors with the $\underset{\sim}{X}_i$'s independent and identically distributed $N(\underset{\sim}{\theta}_1, \sigma_1^2 \underset{\sim}{V}_1)$ and the $\underset{\sim}{Y}_i$'s independent and identically distributed $N(\underset{\sim}{\theta}_2, \sigma_2^2 \underset{\sim}{V}_2)$. Here the θ_i's $\in R^p$ are unknown while the $\underset{\sim}{V}_i$'s are known positive definite matrices, i=1,2. The problem is estimation of $\underset{\sim}{\mu} = \underset{\sim}{\theta}_1 - \underset{\sim}{\theta}_2$. Based on $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ and $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_m$, if $\underset{\sim}{\mu}$ is estimated by $\underset{\sim}{\delta}_{n,m} = \underset{\sim}{\delta}_{n,m}(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n; \underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_m)$ let the incurred loss be given by

$$L(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \underset{\sim}{\delta}_{n,m}) = (\underset{\sim}{\delta}_{n,m} - \underset{\sim}{\mu})' \underset{\sim}{Q} (\underset{\sim}{\delta}_{n,m} - \underset{\sim}{\mu}) + c(n+m), \tag{4.1}$$

where $\underset{\sim}{Q}$ denotes a known positive definite matrix and c(>0) denotes the known cost per sampling unit. The most natural estimator of $\underset{\sim}{\mu}$ is $\underset{\sim}{\bar{W}}_{n,m} = \underset{\sim}{\bar{X}}_n - \underset{\sim}{\bar{Y}}_m$ with risk

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \underset{\sim}{\bar{W}}_{n,m}) \tag{4.2}$$

$$= E_{\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2} [(\bar{\underset{\sim}{W}}_{n,m} - \underset{\sim}{\mu})' \underset{\sim}{Q} (\bar{\underset{\sim}{W}}_{n,m} - \underset{\sim}{\mu})] + c(n+m)$$

$$= E_{\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2} [(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\theta}_1)' \underset{\sim}{Q} (\bar{\underset{\sim}{X}}_n - \underset{\sim}{\theta}_1) + (\bar{\underset{\sim}{Y}}_m - \underset{\sim}{\theta}_2)' \underset{\sim}{Q} (\bar{\underset{\sim}{Y}}_m - \underset{\sim}{\theta}_2)$$

$$+ 2 (\bar{\underset{\sim}{X}}_n - \underset{\sim}{\theta}_1)' \underset{\sim}{Q} (\bar{\underset{\sim}{Y}}_m - \underset{\sim}{\theta}_2)] + c(n+m)$$

$$= \frac{\sigma_1^2}{n} \operatorname{tr} (\underset{\sim}{Q}\ \underset{\sim}{V}_1) + \frac{\sigma_2^2}{m} \operatorname{tr}(\underset{\sim}{Q}\ \underset{\sim}{V}_2) + c(n+m) .$$

When σ_1^2 and σ_2^2 are known, the above risk is minimized at $n = n^* \equiv (\frac{\sigma_1^2}{c} \operatorname{tr}(\underset{\sim}{Q}\ \underset{\sim}{V}_1))^{1/2}$ and $m = m^* \equiv (\frac{\sigma_2^2}{c} \operatorname{tr}(\underset{\sim}{Q}\ \underset{\sim}{V}_2))^{1/2}$. Also, for this pair (n^*,m^*) we have $n^*/m^* = [\frac{\sigma_1^2 \operatorname{tr}(\underset{\sim}{Q}\ \underset{\sim}{V}_1)}{\sigma_2^2 \operatorname{tr}(\underset{\sim}{Q}\ \underset{\sim}{V}_2)}]^{1/2}$. However, when σ_1^2 and σ_2^2 are unknown no fixed sample size will minimize (4.2) simultaneously for all σ_1^2 and σ_2^2. In this case, motivated from the optimal fixed sample size n^* and m^*, and additionally by n^*/m^*, the following sequential procedure is proposed to determine the sample sizes.

$T = N + M$, where

$$N = \inf\{n \geq n_o : (\frac{\hat{s}_n^2}{c} \operatorname{tr}(\underset{\sim}{Q}\ \underset{\sim}{V}_1))^{1/2}\}$$

and (4.3)

$$M = \inf\{m \geq n_o : (\frac{\tilde{s}_m^2}{c} \operatorname{tr}(Q \underset{\sim}{V}_2))^{1/2}\} ,$$

$n_o \geq 2$ and observe $\underset{\sim}{X}_{n+1}$ or $\underset{\sim}{Y}_{m+1}$ if neither process has stopped when

$$n/m \leq [\hat{s}_n^2 \operatorname{tr}(Q \underset{\sim}{V}_1)]^{1/2} / [\tilde{s}_m^2 \operatorname{tr}(Q \underset{\sim}{V}_2)]^{1/2}$$

or (4.4)

$$n/m > [\hat{s}_n^2 \operatorname{tr}(Q \underset{\sim}{V}_1)]^{1/2} / [\tilde{s}_m^2 \operatorname{tr}(Q \underset{\sim}{V}_2)]^{1/2} ,$$

respectively. Here,

$$\hat{s}_n^2 = [(n-1)p]^{-1} \Sigma_{i=1}^{n} (\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}_n)' \underset{\sim}{V}_1^{-1} (\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}_n)$$

and (4.5)

$$\tilde{s}_m^2 = [(m-1)p]^{-1} \Sigma_{j=1}^{m} (\underset{\sim}{Y}_j - \bar{\underset{\sim}{Y}}_m)' \underset{\sim}{V}_2^{-1} (\underset{\sim}{Y}_j - \bar{\underset{\sim}{Y}}_m) ,$$

for every $n,m \geq 2$. The above stopping rule is a multivariate analogue of the stopping rule proposed in Ghosh and Mukhopadhyay (1980). Similar stopping rules were considered by Chou and Hwang (1984) who estimated $\underset{\sim}{\mu}$ by $\bar{\underset{\sim}{X}}_N - \bar{\underset{\sim}{Y}}_M$.

A point of interest concerning the above stopping rule is that if $N=n$ is the first occurrence of $n \geq (\frac{\hat{s}_n^2}{c} \operatorname{tr}(Q \underset{\sim}{V}_1))^{1/2}$ while

$m < \left(\frac{\tilde{s}_m^2}{c}\, tr(Q\, V_2)\right)^{1/2}$ then necessarily we must observe Y_{m+1} since $n/m > [\hat{s}_n^2\, tr(Q\, V_1)]^{1/2} / [\tilde{s}_m^2\, tr(Q\, V_2)]^{1/2}$.

To see how the above sequential sampling procedure can be motivated from a minimax criterion let Θ_i have a $N(\gamma_i, \tau_i^2 B_i)$ prior, i=1,2. Then following the arguments from Section 1.1 the joint posterior distribution of $[\Theta_1', \Theta_2']'$ given $X_i = x_i$, i=1,...,n and $Y_j = y_j$, j=1,...,m is

$$N\left(\begin{bmatrix} \gamma_1 + (I_p + \frac{\sigma_1^2}{n\tau_1^2} V_1 B_1^{-1})^{-1}(\bar{x}_n - \gamma_1) \\ \gamma_2 + (I_p + \frac{\sigma_2^2}{m\tau_2^2} V_2 B_2^{-1})^{-1}(\bar{y}_m - \gamma_2) \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \frac{\sigma_1^2}{n} V_1 (I_p + \frac{\sigma_1^2}{n\tau_1^2} B_1^{-1} V_1)^{-1} & 0 \\ 0 & \frac{\sigma_2^2}{m} V_2 (I_p + \frac{\sigma_2^2}{m\tau_2^2} B_2^{-1} V_2)^{-1} \end{bmatrix} \right).$$

Therefore, the posterior distribution of $\mu = \Theta_1 - \Theta_2$ becomes

$$N(\gamma_1 - \gamma_2 + (I_p + \frac{\sigma_1^2}{n\tau_1^2} V_1 B_1^{-1})^{-1}(\bar{x}_n - \gamma_1) - (I_p + \frac{\sigma_2^2}{m\tau_2^2} V_2 B_2^{-1})^{-1}(\bar{y}_m - \gamma_2),$$

$$\frac{\sigma_1^2}{n} \underset{\sim}{V}_1(\underset{\sim}{I}_p + \frac{\sigma_1^2}{n\tau_1^2} \underset{\sim}{B}_1^{-1}\underset{\sim}{V}_1)^{-1} + \frac{\sigma_2^2}{m} \underset{\sim}{V}_2(\underset{\sim}{I}_p + \frac{\sigma_2^2}{m\tau_2^2} \underset{\sim}{B}_2^{-1}\underset{\sim}{V}_2)^{-1} .$$

Hence, the Bayes estimator of $\underset{\sim}{\mu}$ becomes

$$\underset{\sim}{\delta}^B_{n,m} = (\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2 + (\underset{\sim}{I}_p + \frac{\sigma_1^2}{n\tau_1^2} \underset{\sim}{V}_1\underset{\sim}{B}_1^{-1})^{-1}(\bar{\underset{\sim}{X}}_n-\underset{\sim}{\gamma}_1)-(\underset{\sim}{I}_p + \frac{\sigma_2^2}{m\tau_2^2} \underset{\sim}{V}_2\underset{\sim}{B}_2^{-1})^{-1}(\bar{\underset{\sim}{Y}}_m-\underset{\sim}{\gamma}_2) ,$$

with posterior risk

$$\mathrm{tr}[\underset{\sim}{Q}\{\frac{\sigma_1^2}{n} \underset{\sim}{V}_1(\underset{\sim}{I}_p + \frac{\sigma_1^2}{n\tau_1^2} \underset{\sim}{B}_1^{-1}\underset{\sim}{V}_1)^{-1} + \frac{\sigma_2^2}{m} \underset{\sim}{V}_2(\underset{\sim}{I}_p + \frac{\sigma_2^2}{m\tau_2^2} \underset{\sim}{B}_2^{-1}\underset{\sim}{V}_2)^{-1}\}] + c(n+m) .$$

The posterior risk being independent of any unknown parameters implies that the sequential Bayes rule is a fixed sample rule with sample size determined by minimizing

$$\mathrm{tr}[\underset{\sim}{Q}\{\frac{\sigma_1^2}{n} \underset{\sim}{V}_1(\underset{\sim}{I}_p + \frac{\sigma_1^2}{n\tau_1^2} \underset{\sim}{B}_1^{-1}\underset{\sim}{V}_1)^{-1} + \frac{\sigma_2^2}{m} \underset{\sim}{V}_2(\underset{\sim}{I}_p + \frac{\sigma_2^2}{m\tau_2^2} \underset{\sim}{B}_2^{-1}\underset{\sim}{V}_2)^{-1}\}] + c(n+m)$$

with respect to n and m.

Next, consider the sequence of $\{N(\underset{\sim}{\gamma}_i, l\underset{\sim}{B}_i),\ l \geq 1\}$ priors on $\underset{\sim}{\theta}_i$, i=1,2. Then the corresponding sequence of Bayes estimators for $\underset{\sim}{\mu}$ is

$$\delta^{B}_{\underset{\sim}{n},m} = \underset{\sim}{Y}_1 - \underset{\sim}{Y}_2 + (I_{\underset{\sim}{p}} + \frac{\sigma_1^2}{n1} \underset{\sim}{V}_1 \underset{\sim}{B}_1^{-1})^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{Y}_1) - (I_{\underset{\sim}{p}} + \frac{\sigma_2^2}{m1} \underset{\sim}{V}_2 \underset{\sim}{B}_2^{-1})^{-1}(\bar{\underset{\sim}{Y}}_m - \underset{\sim}{Y}_2) ,$$

with risk

$$\mathrm{tr}[\underset{\sim}{Q}\{\frac{\sigma_1^2}{n} \underset{\sim}{V}_1(I_{\underset{\sim}{p}} + \frac{\sigma_1^2}{n1} \underset{\sim}{B}_1^{-1}\underset{\sim}{V}_1)^{-1} + \frac{\sigma_2^2}{m} \underset{\sim}{V}_2(I_{\underset{\sim}{p}} + \frac{\sigma_2^2}{m1} \underset{\sim}{B}_2^{-1}\underset{\sim}{V}_2)^{-1}\}] + c(n+m) .$$

The latter converges to $K = \frac{\sigma_1^2}{n} \mathrm{tr}(\underset{\sim}{Q}\underset{\sim}{V}_1) + \frac{\sigma_2^2}{m} \mathrm{tr}(\underset{\sim}{Q}\underset{\sim}{V}_2) + c(n+m)$ as $1 \to \infty$, and $R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{\underset{\sim}{W}}_{n,m}) = K$ for all $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$. Hence, under the loss (4.1), the fixed sample rule with sample size determined by minimizing $\frac{\sigma_1^2}{n} \mathrm{tr}(\underset{\sim}{Q}\underset{\sim}{V}_1) + \frac{\sigma_2^2}{m} \mathrm{tr}(\underset{\sim}{Q}\underset{\sim}{V}_2) + c(n+m)$ with respect to n and m, and estimator $\bar{\underset{\sim}{W}}_{n,m}$, is a minimax rule. Consequently, the stopping rule (4.3) can be interpreted as an empirical minimax stopping rule.

In Section 4.2 we show the dominance of a class of estimators over $\bar{\underset{\sim}{W}}_{N,M}$ for the loss (4.1) under the stopping rule (4.3). In Section 4.3 we develop an asymptotic risk expansion for $\bar{\underset{\sim}{W}}_{N,M}$ and for estimators belonging to the above class up to the second order term.

4.2 A Class of James-Stein Estimators Dominating $\bar{\underset{\sim}{W}}_{N,M}$

In this section we consider the class of James-Stein estimators $\delta^{b_1,b_2}_{\underset{\sim}{N},M}(\bar{\underset{\sim}{X}}_N, \bar{\underset{\sim}{Y}}_M)$ where

$$\delta^{b_1,b_2}_{n,m}(\bar{X}_n,\bar{Y}_m) = (\bar{X}_n - \bar{Y}_m) - \Big[\frac{b_1 s''^2_n}{n\|\bar{X}_n - \lambda_1\|^2_{V_1}} Q^{-1}V_1^{-1}(\bar{X}_n - \lambda_1) \qquad (4.6)$$

$$- \frac{b_2 s''^2_m}{m\|\bar{Y}_m - \lambda_2\|^2_{V_2}} Q^{-1}V_2^{-1}(\bar{Y}_m - \lambda_2)\Big]$$

for every n, $m \geq n_0$ (≥ 2), where $s''^2_n = \frac{p}{p+2}\hat{s}^2_n$, $s''^2_m = \frac{p}{p+2}\tilde{s}^2_m$, b_i, $i=1,2$ are constants, $\|\bar{X}_n - \lambda_1\|^2_{V_1} = (\bar{X}_n - \lambda_1)'V_1^{-1}Q^{-1}V_1^{-1}(\bar{X}_n - \lambda_1)$, $\|\bar{Y}_m - \lambda_2\|^2_{V_2} = (\bar{Y}_m - \lambda_2)'V_2^{-1}Q^{-1}V_2^{-1}(\bar{Y}_m - \lambda_2)$, and λ_i, $i=1,2$ are the known points towards which we shrink $\bar{X}_n$ and $\bar{Y}_m$, respectively. The main result of this section is as follows:

Theorem 4.1 Under the stopping rule (4.3) and the loss (4.1),

$$R(\theta_1,\theta_2,\sigma_1^2,\sigma_2^2,\delta^{b_1,b_2}_{N,M}) < R(\theta_1,\theta_2,\sigma_1^2,\sigma_2^2,\bar{W}_{N,M})$$

for every $b_1,b_2 \in (0,p-2)$.

Proof: Note that

$$R(\theta_1,\theta_2,\sigma_1^2,\sigma_2^2,\delta^{b_1,b_2}_{N,M}) - R(\theta_1,\theta_2,\sigma_1^2,\sigma_2^2,\bar{W}_{N,M}) \qquad (4.7)$$

$$= -2b_1 E_{\theta_1,\sigma_1^2}\Big[\frac{s''^2_N}{N\|\bar{X}_N - \lambda_1\|^2_{V_1}}(\bar{X}_N - \lambda_1)'V_1^{-1}(\bar{X}_N - \theta_1)\Big]$$

$$-2b_2 E_{\underset{\sim}{\theta_2}, \sigma_2^2}\left[\frac{\bar{s}_M^{-2}}{M\|\bar{Y}_M - \underset{\sim}{\lambda_2}\|_{V_2}^2}(\bar{Y}_M - \underset{\sim}{\lambda_2})' V_2^{-1}(\bar{Y}_M - \underset{\sim}{\theta_2})\right]$$

$$+ b_1^2 E_{\underset{\sim}{\theta_1}, \sigma_1^2}\left[\frac{\bar{s}_N^{-4}}{N^2\|\bar{X}_N - \underset{\sim}{\lambda_1}\|_{V_1}^2}\right] + b_2^2 E_{\underset{\sim}{\theta_2}, \sigma_2^2}\left[\frac{\bar{s}_M^{-4}}{M^2\|\bar{Y}_M - \underset{\sim}{\lambda_2}\|_{V_2}^2}\right]$$

$$- 2b_1 b_2 E_{\underset{\sim}{\theta_1}, \underset{\sim}{\theta_2}, \sigma_1^2, \sigma_2^2}\left[\frac{\bar{s}_N^{-2}}{N\|\bar{X}_N - \underset{\sim}{\lambda_1}\|_{V_1}^2} \cdot \frac{\bar{s}_M^{-2}}{M\|\bar{Y}_M - \underset{\sim}{\lambda_2}\|_{V_2}^2}\right.$$

$$\left. \cdot (\bar{X}_N - \underset{\sim}{\lambda_1})' V_1^{-1} Q^{-1} V_2^{-1}(\bar{Y}_M - \underset{\sim}{\lambda_2})\right] .$$

Here, by using Stein's identity as in the proof of Theorem 2.1 we have

$$E_{\underset{\sim}{\theta_1}, \sigma_1^2}\left[\frac{\bar{s}_N^{-2}}{N\|\bar{X}_N - \underset{\sim}{\lambda_1}\|_{V_1}^2}(\bar{X}_N - \underset{\sim}{\lambda_1})' V_1^{-1}(\bar{X}_N - \underset{\sim}{\theta_1})\right] \tag{4.8}$$

$$= \sigma_1^2(p-2) E_{\underset{\sim}{\theta_1}, \sigma_1^2}\left[\frac{\bar{s}_N^{-2}}{N^2\|\bar{X}_N - \underset{\sim}{\lambda_1}\|_{V_1}^2}\right] .$$

Similarly,

$$E_{\underset{\sim}{\theta_2}, \sigma_2^2}\left[\frac{\bar{s}_M^{-2}}{M\|\bar{Y}_M - \underset{\sim}{\lambda_2}\|_{V_2}^2}(\bar{Y}_M - \underset{\sim}{\lambda_2})' V_2^{-1}(\bar{Y}_M - \underset{\sim}{\theta_2})\right] \tag{4.9}$$

$$= \sigma_2^2(p-2)E_{\underset{\sim}{\theta}_2, \sigma_2^2}\left[\frac{\bar{s}_M^{-2}}{M^2 \|\bar{Y}_{\underset{\sim}{M}} - \underset{\sim}{\lambda}_2\|_{V_2}^2}\right] .$$

Therefore, using (4.8) and (4.9)

rhs of (4.7) (4.10)

$$= -2\sigma_1^2(p-2)b_1 E_{\underset{\sim}{\theta}_1, \sigma_1^2}\left[\frac{\dot{s}_N^{-2}}{N^2 \|\bar{X}_{\underset{\sim}{N}} - \underset{\sim}{\lambda}_1\|_{V_1}^2}\right] - 2\sigma_2^2(p-2)b_2 E_{\underset{\sim}{\theta}_2, \sigma_2^2}\left[\frac{\bar{s}_M^{-2}}{M^2 \|\bar{Y}_{\underset{\sim}{M}} - \underset{\sim}{\lambda}_2\|_{V_2}^2}\right]$$

$$+ b_1^2 E_{\underset{\sim}{\theta}_1, \sigma_1^2}\left[\frac{\dot{s}_N^{-4}}{N^2 \|\bar{X}_{\underset{\sim}{N}} - \underset{\sim}{\lambda}_1\|_{V_1}^2}\right] + b_2^2 E_{\underset{\sim}{\theta}_2, \sigma_2^2}\left[\frac{\bar{s}_M^{-4}}{M^2 \|\bar{Y}_{\underset{\sim}{M}} - \underset{\sim}{\lambda}_2\|_{V_2}^2}\right]$$

$$- 2b_1 b_2 E_{\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2}\left[\frac{\dot{s}_N^{-2}}{N \|\bar{X}_{\underset{\sim}{N}} - \underset{\sim}{\lambda}_1\|_{V_1}^2} \cdot \frac{\bar{s}_M^{-2}}{M \|\bar{Y}_{\underset{\sim}{M}} - \underset{\sim}{\lambda}_2\|_{V_2}^2}\right.$$

$$\left. \cdot (\bar{X}_{\underset{\sim}{N}} - \underset{\sim}{\lambda}_1)' V_1^{-1} \underset{\sim}{Q}^{-1} V_2^{-1} (\bar{Y}_{\underset{\sim}{M}} - \underset{\sim}{\lambda}_2)\right] .$$

From the Cauchy-Schwarz inequality we have for any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $|\underset{\sim}{a}'\underset{\sim}{b}| \leq \|\underset{\sim}{a}\| \|\underset{\sim}{b}\|$, where $\|.\|$ denotes the usual Euclidean norm. Then, using the additional fact that there exists a nonsingular matrix $\underset{\sim}{P}$ such that $\underset{\sim}{P}\,\underset{\sim}{Q}^{-1}\underset{\sim}{P}' = \underset{\sim}{I}_p$, we have

$$- \frac{\dot{s}_N^{-2}}{N \|\bar{X}_{\underset{\sim}{N}} - \underset{\sim}{\lambda}_1\|_{V_1}^2} \cdot \frac{\bar{s}_M^{-2}}{M \|\bar{Y}_{\underset{\sim}{M}} - \underset{\sim}{\lambda}_2\|_{V_2}^2} (\bar{X}_{\underset{\sim}{N}} - \underset{\sim}{\lambda}_1)' V_1^{-1} \underset{\sim}{Q}^{-1} V_2^{-1} (\bar{Y}_{\underset{\sim}{M}} - \underset{\sim}{\lambda}_2) \qquad (4.11)$$

$$= -\left[\frac{\ddot{s}_N^2}{N\|\bar{X}_{\sim N} - \lambda_{\sim 1}\|^2_{V_1}} P'^{-1} V_1^{-1}(\bar{X}_{\sim N} - \lambda_{\sim 1})\right]'\left[\frac{\bar{s}_M^2}{M\|\bar{Y}_{\sim M} - \lambda_{\sim 2}\|^2_{V_2}} P'^{-1} V_2^{-1}(\bar{Y}_{\sim M} - \lambda_{\sim 2})\right]$$

$$\leq \left[\frac{\ddot{s}_N^2}{N\|\bar{X}_{\sim N} - \lambda_{\sim 1}\|^2_{V_1}}\right]\left[\frac{\bar{s}_M^2}{M\|\bar{Y}_{\sim M} - \lambda_{\sim 2}\|^2_{V_2}}\right] .$$

Consequently, using (4.11)

rhs of (4.10) (4.12)

$$\leq -2\sigma_1^2(p-2)b_1 E_{\theta_{\sim 1}, \sigma_1^2}\left[\frac{\ddot{s}_N^2}{N^2\|\bar{X}_{\sim N} - \lambda_{\sim 1}\|^2_{V_1}}\right] - 2\sigma_2^2(p-2)b_2 E_{\theta_{\sim 2}, \sigma_2^2}\left[\frac{\bar{s}_M^2}{M^2\|\bar{Y}_{\sim M} - \lambda_{\sim 2}\|^2_{V_2}}\right]$$

$$+ b_1^2 E_{\theta_{\sim 1}, \sigma_1^2}\left[\frac{\ddot{s}_N^4}{N^2\|\bar{X}_{\sim N} - \lambda_{\sim 1}\|^2_{V_1}}\right] + b_2^2 E_{\theta_{\sim 2}, \sigma_2^2}\left[\frac{\bar{s}_M^4}{M^2\|\bar{Y}_{\sim M} - \lambda_{\sim 2}\|^2_{V_2}}\right]$$

$$+ 2b_1 b_2 E_{\theta_{\sim 1}, \theta_{\sim 2}, \sigma_1^2, \sigma_2^2}\left[\frac{\ddot{s}_N^2}{N\|\bar{X}_{\sim N} - \lambda_{\sim 1}\|^2_{V_1}} \cdot \frac{\bar{s}_M^2}{M\|\bar{Y}_{\sim M} - \lambda_{\sim 2}\|^2_{V_2}}\right]$$

$$= -2\sigma_1^2(p-2)b_1 E_{\theta_{\sim 1}, \sigma_1^2}\left[\frac{\ddot{s}_N^2}{N^2\|\bar{X}_{\sim N} - \lambda_{\sim 1}\|^2_{V_1}}\right] - 2\sigma_2^2(p-2)b_2 E_{\theta_{\sim 2}, \sigma_2^2}\left[\frac{\bar{s}_M^2}{M^2\|\bar{Y}_{\sim M} - \lambda_{\sim 2}\|^2_{V_2}}\right]$$

$$+ E_{\theta_{\sim 1}, \theta_{\sim 2}, \sigma_1^2, \sigma_2^2}\left[\left\{\frac{b_1\ddot{s}_N^2}{N\|\bar{X}_{\sim N} - \lambda_{\sim 1}\|^2_{V_1}} + \frac{b_2\bar{s}_M^2}{M\|\bar{Y}_{\sim M} - \lambda_{\sim 2}\|^2_{V_2}}\right\}^2\right] .$$

Using the fact that $(a+b)^r \leq 2^{r-1}(a^r+b^r)$, for $a>0$, $b>0$ and $r\geq 1$ then

$$E_{\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2}\left[\left\{\frac{b_1 \ddot{s}_N^2}{N\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}} + \frac{b_2 \bar{s}_M^2}{M\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}}\right\}^2\right] \tag{4.13}$$

$$\leq 2b_1^2 E_{\underset{\sim}{\theta}_1,\sigma_1^2}\left[\frac{\ddot{s}_N^4}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|^2_{V_1}}\right] + 2b_2^2 E_{\underset{\sim}{\theta}_2,\sigma_2^2}\left[\frac{\bar{s}_M^4}{M^2\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|^2_{V_2}}\right].$$

Therefore, using (4.13)

extreme rhs of (4.12) (4.14)

$$\leq -2\sigma_1^2(p-2)b_1 E_{\underset{\sim}{\theta}_1,\sigma_1^2}\left[\frac{\ddot{s}_N^2}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|^2_{V_1}}\right] - 2\sigma_2^2(p-2)b_2 E_{\underset{\sim}{\theta}_2,\sigma_2^2}\left[\frac{\bar{s}_M^2}{M^2\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|^2_{V_2}}\right]$$

$$+ 2b_1^2 E_{\underset{\sim}{\theta}_1,\sigma_1^2}\left[\frac{\ddot{s}_N^4}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|^2_{V_1}}\right] + 2b_2^2 E_{\underset{\sim}{\theta}_2,\sigma_2^2}\left[\frac{\bar{s}_M^4}{M^2\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|^2_{V_2}}\right].$$

From the proof of Theorem 2.1 we have

$$E_{\underset{\sim}{\theta}_1,\sigma_1^2}\left[\frac{\sigma_1^2 \ddot{s}_N^2}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|^2_{V_1}}\right] \tag{4.15}$$

$$= \Sigma_{n=n_0}^{\infty} E_{\underset{\sim}{\theta}_1, \sigma_1^2}\left[\frac{\sigma_1^2 \ddot{s}_n^2}{n^2 \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}_1\|_{V_1}^2} I_{[N=n]}\right]$$

$$= \Sigma_{n=n_0}^{\infty} E_{\sigma_1^2}\left[\frac{\ddot{s}_n^2}{n} I_{[N=n]}\right] E_{\underset{\sim}{\theta}_1, \sigma_1^2}\left[\frac{\sigma_1^2}{n \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}_1\|_{V_1}^2}\right]$$

$$= \Sigma_{n=n_0}^{\infty} E_{\sigma_1^2}\left[\frac{\ddot{s}_n^2}{n} I_{[N=n]}\right] g_p(\Delta_n)$$

$$= E_{\sigma_1^2}\left[g_p(\Delta_N) \frac{\ddot{s}_N^2}{N}\right] .$$

Similarly,

$$E_{\underset{\sim}{\theta}_1, \sigma_1^2}\left[\frac{\sigma_1^2 \ddot{s}_N^4}{N^2 \|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}^2}\right] = E_{\sigma_1^2}\left[g_p(\Delta_N) \frac{\ddot{s}_N^4}{N}\right] , \tag{4.16}$$

$$E_{\underset{\sim}{\theta}_2, \sigma_2^2}\left[\frac{\sigma_2^2 \bar{s}_M^2}{M^2 \|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}^2}\right] = E_{\sigma_2^2}\left[g_p(\Delta_M) \frac{\bar{s}_M^2}{M}\right] , \tag{4.17}$$

and

$$E_{\underset{\sim}{\theta}_2, \sigma_2^2}\left[\frac{\sigma_2^2 \bar{s}_M^4}{M^2 \|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}^2}\right] = E_{\sigma_2^2}\left[g_p(\Delta_M) \frac{\bar{s}_M^4}{M}\right] , \tag{4.18}$$

where $\Delta_n = \Delta_n(\underset{\sim}{\theta}_1, \sigma_1^2)$ and $\Delta_m = \Delta_m(\underset{\sim}{\theta}_2, \sigma_2^2)$. Therefore, from (4.15) - (4.18)

$$\text{rhs of (4.14)} = 2b_1 E_{\sigma_1^2}[g_p(\Delta_N)\frac{\ddot{s}_N^2}{N}(b_1\frac{\ddot{s}_N^2}{\sigma_1^2} - (p-2))] \tag{4.19}$$

$$+ 2b_2 E_{\sigma_2^2}[g_p(\Delta_M)\frac{\bar{s}_M^2}{M}(b_2\frac{\bar{s}_M^2}{\sigma_2^2} - (p-2))] .$$

Since $b_1, b_2 \in (0, p-2)$ then

$$\text{rhs of (4.19)} < 2(p-2)b_1 E_{\sigma_1^2}[g_p(\Delta_N)\frac{\ddot{s}_N^2}{N}(\frac{\ddot{s}_N^2}{\sigma_1^2} - 1)] \tag{4.20}$$

$$+ 2(p-2)b_2 E_{\sigma_2^2}[g_p(\Delta_M)\frac{\bar{s}_M^2}{M}(\frac{\bar{s}_M^2}{\sigma_2^2} - 1)] .$$

Consequently using (4.7), (4.10), (4.12), (4.19) and (4.20)

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \underset{\sim}{\delta}_{N,M}^{b_1,b_2}) - R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{\underset{\sim}{W}}_{N,M}) \tag{4.21}$$

$$< 2(p-2)b_1 E_{\sigma_1^2}[g_p(\Delta_N)\frac{\ddot{s}_N^2}{N}(\frac{\ddot{s}_N^2}{\sigma_1^2} - 1)]$$

$$+ 2(p-2)b_2 E_{\sigma_2^2}[g_p(\Delta_M)\frac{\bar{s}_M^2}{M}(\frac{\bar{s}_M^2}{\sigma_2^2} - 1)] .$$

By applying the latter half of the method of proof for Theorem 2.1 to each term in the right hand side of (4.21) the result follows.

4.3 Asymptotic Risk Expansion for $\bar{W}_{N,M}$ and $\delta^{b_1,b_2}_{N,M}$

In this section we first obtain an asymptotic risk expansion up to the second order term for $\bar{W}_{N,M}$ (trivial) and then for $\delta^{b_1,b_2}_{N,M}$.

Observe that

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{W}_{N,M}) \tag{4.22}$$

$$= E_{\underset{\sim}{\theta}_1, \sigma_1^2}[(\bar{X}_N - \underset{\sim}{\theta}_1)'Q(\bar{X}_N - \underset{\sim}{\theta}_1)] + c\, E_{\sigma_1^2}[N]$$

$$+ E_{\underset{\sim}{\theta}_2, \sigma_2^2}[(\bar{Y}_M - \underset{\sim}{\theta}_2)'Q(\bar{Y}_M - \underset{\sim}{\theta}_2)] + c\, E_{\sigma_2^2}[M]$$

$$= \sigma_1^2\, \mathrm{tr}(QV_1)\, E_{\sigma_1^2}[N^{-1}] + c\, E_{\sigma_1^2}[N]$$

$$+ \sigma_2^2\, \mathrm{tr}(QV_2)\, E_{\sigma_2^2}[M^{-1}] + c\, E_{\sigma_2^2}[M]$$

$$= 2c^{1/2}(\sigma_1^2\, \mathrm{tr}(QV_1))^{1/2} + c\, E_{\sigma_1^2}\left[\frac{(N-n^*)^2}{N}\right]$$

$$+ 2c^{1/2}(\sigma_2^2\, \mathrm{tr}(QV_2))^{1/2} + c\, E_{\sigma_2^2}\left[\frac{(M-m^*)^2}{M}\right].$$

Following the arguments of Section 2.3 concerning the asymptotic risk expansion of $\bar{X}_N$ we have as $c \to 0$,

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{\underset{\sim}{W}}_{N,M}) \tag{4.23}$$

$$= 2c^{1/2} (\sigma_1^2 \operatorname{tr}(\underset{\sim}{Q}\underset{\sim}{V}_1))^{1/2} + \frac{c}{2p} + 2c^{1/2} (\sigma_2^2 \operatorname{tr}(\underset{\sim}{Q}\underset{\sim}{V}_2))^{1/2} + \frac{c}{2p} + o(c)$$

$$= 2c^{1/2} [(\sigma_1^2 \operatorname{tr}(\underset{\sim}{Q}\underset{\sim}{V}_1))^{1/2} + (\sigma_2^2 \operatorname{tr}(\underset{\sim}{Q}\underset{\sim}{V}_2))^{1/2}] + \frac{c}{p} + o(c) .$$

Next, we find the asymptotic risk expansion for $\underset{\sim}{\delta}_{N,M}^{b_1,b_2}$. From (4.7) - (4.10) and (4.15) - (4.18) it follows that

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \underset{\sim}{\delta}_{N,M}^{b_1,b_2}) \tag{4.24}$$

$$= R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{\underset{\sim}{W}}_{N,M}) + \frac{b_1^2}{\sigma_1^2} E_{\sigma_1^2}[g_p(\Delta_N) \frac{\ddot{s}_N^4}{N}]$$

$$- 2(p-2)b_1 E_{\sigma_1^2}[g_p(\Delta_N) \frac{\ddot{s}_N^2}{N}] + \frac{b_2^2}{\sigma_2^2} E_{\sigma_2^2}[g_p(\Delta_M) \frac{\bar{s}_M^4}{M}]$$

$$-2(p-2)b_2 E_{\sigma_2^2}[g_p(\Delta_M) \frac{\bar{s}_M^2}{M}]$$

$$- 2b_1 b_2 E_{\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2}[\frac{\ddot{s}_N^2}{N \| \bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1 \|_{\underset{\sim}{V}_1}^2} \cdot \frac{\bar{s}_M^2}{M \| \bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2 \|_{\underset{\sim}{V}_2}^2}$$

$$\cdot (\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1)' \underset{\sim}{V}_1^{-1} \underset{\sim}{Q} \underset{\sim}{V}_2^{-1} (\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2)] .$$

Now, for technical reasons we shall handle the remainder of the risk expansion separately for the four cases (i) $\underset{\sim}{\theta}_1 = \underset{\sim}{\lambda}_1$, $\underset{\sim}{\theta}_2 = \underset{\sim}{\lambda}_2$, (ii) $\underset{\sim}{\theta}_1 \neq \underset{\sim}{\lambda}_1$, $\underset{\sim}{\theta}_2 \neq \underset{\sim}{\lambda}_2$, (iii) $\underset{\sim}{\theta}_1 = \underset{\sim}{\lambda}_1$, $\underset{\sim}{\theta}_2 \neq \underset{\sim}{\lambda}_2$ and (iv) $\underset{\sim}{\theta}_1 \neq \underset{\sim}{\lambda}_1$, $\underset{\sim}{\theta}_2 = \underset{\sim}{\lambda}_2$. Since case (iii) and (iv) are essentially the same we shall only explore case (iii). However, using results of Section 2.3 we have, respectively, for $\underset{\sim}{\theta}_i = \underset{\sim}{\lambda}_i$, i=1,2, as $c \to 0$,

$$c^{-1/2}\left[\frac{b_1^2}{\sigma_1^2} E_{\sigma_1^2}\left(g_p(\Delta_N)\frac{\ddot{s}_N^4}{N}\right) - 2(p-2)b_1 E_{\sigma_1^2}\left(g_p(\Delta_N)\frac{\ddot{s}_N^2}{N}\right)\right] \tag{4.25}$$

$$\to -\sigma_1 b_1\left(\frac{p}{p+2}\right)^2(2(p^2-4)p^{-1} - b_1)(\operatorname{tr} \underset{\sim}{Q}\underset{\sim}{V}_1)^{-1/2} K_1,$$

and

$$c^{-1/2}\left[\frac{b_2^2}{\sigma_2^2} E_{\sigma_2^2}\left(g_p(\Delta_M)\frac{\ddot{s}_M^4}{M}\right) - 2(p-2)b_2 E_{\sigma_2^2}\left(g_p(\Delta_M)\frac{\ddot{s}_M^2}{M}\right)\right] \tag{4.26}$$

$$\to -\sigma_2 b_2\left(\frac{p}{p+2}\right)^2(2(p^2-4)p^{-1} - b_2)(\operatorname{tr} \underset{\sim}{Q}\underset{\sim}{V}_2)^{-1/2} K_2,$$

where $K_1 = \int_0^\infty \prod_{j=1}^{p} (1 + 2ta_{1j}^{-1})^{-1/2}\, dt$, $K_2 = \int_0^\infty \prod_{j=1}^{p} (1 + 2ta_{2j}^{-1})^{-1/2}\, dt$, and the a_{ij}'s, j=1,...,p, are the diagonal elements of $\underset{\sim}{A}_i = \underset{\sim}{D}_i \underset{\sim}{V}_i \underset{\sim}{D}_i'$, i=1,2 (here, $\underset{\sim}{D}_i$ is a matrix such that $\underset{\sim}{D}_i \underset{\sim}{Q}^{-1} \underset{\sim}{D}_i' = \underset{\sim}{I}_p$ and $\underset{\sim}{D}_i \underset{\sim}{V}_i \underset{\sim}{D}_i' = \underset{\sim}{A}_i$, diagonal, i=1,2). Also from Section 2.3, we have, respectively, for $\underset{\sim}{\theta}_i \neq \underset{\sim}{\lambda}_i$, i=1,2, as $c \to 0$,

$$c^{-1}\left[\frac{b_1^2}{\sigma_1^2} E_{\sigma_1^2}(g_p(\Delta_N)\frac{\ddot{s}_N^4}{N}) - 2(p-2)b_1 E_{\sigma_1^2}(g_p(\Delta_N)\frac{\ddot{s}_N^2}{N})\right] \tag{4.27}$$

$$\to - b_1(\frac{p}{p+2})^2(2(p^2-4)p^{-1} - b_1)\sigma_1^2(\text{tr } \underset{\sim}{QV}_1)^{-1}\|\underset{\sim}{\Theta}_1 - \underset{\sim}{\lambda}_1\|_{V_1}^{-2}$$

and

$$c^{-1}\left[\frac{b_2^2}{\sigma_2^2} E_{\sigma_2^2}(g_p(\Delta_M)\frac{\bar{s}_M^4}{M}) - 2(p-2)b_2 E_{\sigma_2^2}(g_p(\Delta_M)\frac{\bar{s}_M^2}{M})\right] \tag{4.28}$$

$$\to - b_2(\frac{p}{p+2})^2(2(p^2-4)p^{-1} - b_2)\sigma_2^2(\text{tr } \underset{\sim}{QV}_2)^{-1}\|\underset{\sim}{\Theta}_2 - \underset{\sim}{\lambda}_2\|_{V_2}^{-2}.$$

Consequently, we need only concern ourselves with the last term in the right hand side of (4.24) for the three cases we will consider. However, we shall further require use of results form Section 2.3 such as $\ddot{s}_N^2 \xrightarrow{a.s.} \frac{p}{p+2}\sigma_1^2$, $\bar{s}_M^2 \xrightarrow{a.s.} \frac{p}{p+2}\sigma_2^2$, $Nc^{1/2} \xrightarrow{a.s.} (\sigma_1^2 \text{ tr}(\underset{\sim}{QV}_1))^{1/2}$ $Mc^{1/2} \xrightarrow{a.s.} (\sigma_2^2 \text{ tr}(\underset{\sim}{QV}_2))^{1/2}$.

For case (i) $\underset{\sim}{\Theta}_1 = \underset{\sim}{\lambda}_1$, $\underset{\sim}{\Theta}_2 = \underset{\sim}{\lambda}_2$ let $\underset{\sim}{B}_{N1} = \sqrt{N}(\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1)$ and $\underset{\sim}{B}_{M2} = \sqrt{M}(\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2)$. Then as $c \to 0$, using Anscombe's Theorem,

$$\begin{bmatrix} \underset{\sim}{B}_{N1} \\ \underset{\sim}{B}_{M2} \end{bmatrix} \xrightarrow{L} N\left(\begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \end{bmatrix}, \begin{bmatrix} \sigma_1^2\underset{\sim}{V}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \sigma_2^2\underset{\sim}{V}_2 \end{bmatrix}\right), \tag{4.29}$$

implying

$$\begin{bmatrix} -B_{N1} \\ B_{M2} \end{bmatrix} \xrightarrow{L} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 V_1 & 0 \\ 0 & \sigma_2^2 V_2 \end{bmatrix} \right) . \tag{4.30}$$

Hence, from (4.29) and (4.30)

$$\frac{B_{N1}' V_1^{-1} Q^{-1} V_2^{-1} B_{M2}}{\|B_{N1}\|_{V_1}^2 \|B_{M2}\|_{V_2}^2} \xrightarrow{L} Z \quad \text{(say)} \tag{4.31}$$

and

$$\frac{-B_{N1}' V_1^{-1} Q^{-1} V_2^{-1} B_{M2}}{\|B_{N1}\|_{V_1}^2 \|B_{M2}\|_{V_2}^2} \xrightarrow{L} Z . \tag{4.32}$$

Since $E|Z| < \infty$, from (4.31) and (4.32), $E[Z] = 0$. Consequently, as $c \to 0$,

$$c^{-1/2} \frac{\dot{s}_N^2}{N \|\bar{X}_N - \lambda_1\|_{V_1}^2} \cdot \frac{\bar{s}_M^2}{M \|\bar{Y}_M - \lambda_2\|_{V_2}^2} (\bar{X}_N - \lambda_1)' V_1^{-1} Q^{-1} V_2^{-1} (\bar{Y}_M - \lambda_2) \tag{4.33}$$

$$= \frac{\dot{s}_N^2}{(Nc)^{1/2}} \frac{\bar{s}_M^2}{(Mc)^{1/2}} \frac{B_1' V_1^{-1} Q^{-1} V_2^{-1} B_2}{\|B_{N1}\|_{V_1}^2 \|B_{M2}\|_{V_2}^2}$$

$$\xrightarrow{L} \left(\frac{p}{p+2}\right)^2 \frac{\sigma_1^2}{(\sigma_1^2 \operatorname{tr}(QV_1))^{1/4}} \frac{\sigma_2^2}{(\sigma_2^2 \operatorname{tr}(QV_2))^{1/4}} Z .$$

Next we prove the uniform integrability (in $c \le c_0$) of the left hand side of (4.33) for case (i). Again using the Cauchy-Schwarz inequality, with probability one,

$$|c^{-1/2} \frac{\ddot{s}_N^2}{N\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}^2} \cdot \frac{\bar{s}_M^2}{M\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}^2} (\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1)' V_1^{-1} Q^{-1} V_2^{-1} (\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2)| \qquad (4.34)$$

$$\le c^{-1/2} \frac{\ddot{s}_N^2}{N\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}} \cdot \frac{\bar{s}_M^2}{M\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}} .$$

Thus it suffices to prove the uniform integrability (in $c \le c_0$) of the right hand side of (4.34). With this end note that

$$E_{\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2} [c^{-1/2} \frac{\ddot{s}_N^2}{N\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}} \frac{\bar{s}_M^2}{M\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}} I_{[c^{-1/2} \frac{\ddot{s}_N^2}{N\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}} \frac{\bar{s}_M^2}{M\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}} > d]}] \qquad (4.35)$$

$$\le d^{-1} E_{\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2} [c^{-1} \frac{\ddot{s}_N^4}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}^2} \frac{\bar{s}_M^4}{M^2\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}^2}] .$$

By previous arguments

$$E_{\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2} [\frac{\ddot{s}_N^4}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}^2} \frac{\bar{s}_M^4}{M^2\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}^2}] \qquad (4.36)$$

$$= \sum_{n=n_o}^{\infty}\sum_{m=n_o}^{\infty} E_{\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2}\left[\frac{\ddot{s}_n^4}{n^2\|\bar{\underset{\sim}{X}}_n-\underset{\sim}{\lambda}_1\|_{\underset{\sim}{V}_1}^2}\,\frac{\bar{s}_m^4}{m^2\|\bar{\underset{\sim}{Y}}_m-\underset{\sim}{\lambda}_2\|_{\underset{\sim}{V}_2}^2}\, I_{[N=n,M=m]}\right]$$

$$= \sum_{n=n_o}^{\infty}\sum_{m=n_o}^{\infty} E_{\sigma_1^2,\sigma_2^2}\left[\frac{\ddot{s}_n^4}{\sigma_{1n}^2}\,\frac{\bar{s}_m^4}{\sigma_{2m}^2}\, I_{[N=n,M=m]}\right] K_1K_2$$

$$= \frac{K_1}{\sigma_1^2}\,\frac{K_2}{\sigma_2^2}\, E_{\sigma_1^2,\sigma_2^2}\left[\frac{\ddot{s}_N^4}{N}\cdot\frac{\bar{s}_M^4}{M}\right] .$$

Then, using Holder's inequality and (4.36)

$$\text{lhs of (4.35)} \le \frac{K_1}{\sigma_1^2}\,\frac{K_2}{\sigma_2^2}\, d^{-1} E_{\sigma_1^2}^{1/2}\left[\frac{\ddot{s}_N^8}{cN^2}\right] E_{\sigma_2^2}^{1/2}\left[\frac{\bar{s}_M^8}{cM^2}\right] . \tag{4.37}$$

By following the proof of Theorem 2.1 for $n_o \ge 2$, $c \le c_o < 1$, $E_{\sigma_1^2}\left[\frac{\ddot{s}_N^8}{cN^2}\right]$ and $E_{\sigma_2^2}\left[\frac{\bar{s}_M^8}{cM^2}\right]$ can be bounded by a constant (depending on c_o) uniformly in $c \le c_o$. Combining (4.34) - (4.37) the uniform integrability of the left hand side of (4.33) follows. Thus, for $n_o \ge 2$, case (i), as $c \to 0$,

$$c^{-1/2} E_{\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2}\left[\frac{\ddot{s}_N^2}{N\|\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}_1\|_{\underset{\sim}{V}_1}^2}\,\frac{\bar{s}_M^2}{M\|\bar{\underset{\sim}{Y}}_M-\underset{\sim}{\lambda}_2\|_{\underset{\sim}{V}_2}^2}\,(\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}_1)'\underset{\sim}{V}_1^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}_2^{-1}(\bar{\underset{\sim}{Y}}_M-\underset{\sim}{\lambda}_2)\right]$$

(4.38)

$$+ \left(\frac{p}{p+2}\right)^2 \frac{\sigma_1^2}{(\sigma_1^2 \operatorname{tr}(\underset{\sim}{QV}_1))^{1/4}} \frac{\sigma_2^2}{(\sigma_2^2 \operatorname{tr}(\underset{\sim}{QV}_2))^{1/4}} E[Z]$$

$$= 0 .$$

Consequently, for $n_o \geq 2$, case (i), as $c \to 0$,

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \underset{\sim}{\delta}^{b_1,b_2}_{N,M}) \tag{4.39}$$

$$= 2c^{1/2}[(\sigma_1^2 \operatorname{tr}(\underset{\sim}{QV}_1))^{1/2} + (\sigma_2^2 \operatorname{tr}(\underset{\sim}{QV}_2))^{1/2}]$$

$$- c^{1/2}[\sigma_1 b_1 \left(\frac{p}{p+2}\right)^2 (2(p^2-4)p^{-1} - b_1)(\operatorname{tr} \underset{\sim}{QV}_1)^{-1/2} K_1$$

$$+ \sigma_2 b_2 \left(\frac{p}{p+2}\right)^2 (2(p^2-4)p^{-1} - b_2)(\operatorname{tr} \underset{\sim}{QV}_2)^{-1/2} K_2] + o(c^{1/2}) .$$

Thus, in this case,

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{\underset{\sim}{W}}_{N,M}) - R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \underset{\sim}{\delta}^{b_1,b_2}_{N,M}) \tag{4.40}$$

$$= c^{1/2}[\sigma_1 b_1 \left(\frac{p}{p+2}\right)^2 (2(p^2-4)p^{-1} - b_1)(\operatorname{tr} \underset{\sim}{QV}_1)^{-1/2} K_1$$

$$+ \sigma_2 b_2 \left(\frac{p}{p+2}\right)^2 (2(p^2-4)p^{-1} - b_2)(\operatorname{tr} \underset{\sim}{QV}_2)^{-1/2} K_2] + o(c^{1/2}) .$$

For case (ii) $\underset{\sim}{\theta}_1 \neq \underset{\sim}{\lambda}_1$, $\underset{\sim}{\theta}_2 \neq \underset{\sim}{\lambda}_2$ using the additional fact that as $c \to 0$, $\bar{\underset{\sim}{X}}_N \xrightarrow{a.s.} \underset{\sim}{\theta}_1$, $\bar{\underset{\sim}{Y}}_M \xrightarrow{a.s.} \underset{\sim}{\theta}_2$, one gets

$$c^{-1} \frac{\ddot{s}_N^2}{N\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}^2} \cdot \frac{\bar{s}_M^{-2}}{M\|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}^2} (\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1)' \underset{\sim}{V}_1^{-1} \underset{\sim}{Q}^{-1} \underset{\sim}{V}_2^{-1} (\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2) \qquad (4.41)$$

$$\xrightarrow{P} \left(\frac{p}{p+2}\right)^2 \frac{\sigma_1^2}{(\sigma_1^2 \operatorname{tr}(\underset{\sim}{Q}\underset{\sim}{V}_1))^{1/2}} \frac{1}{\|\underset{\sim}{\theta}_1 - \underset{\sim}{\lambda}_1\|_{V_1}^2} \frac{\sigma_2^2}{(\sigma_2^2 \operatorname{tr}(\underset{\sim}{Q}\underset{\sim}{V}_2))^{1/2}} \frac{1}{\|\underset{\sim}{\theta}_2 - \underset{\sim}{\lambda}_2\|_{V_2}^2}$$

$$\cdot (\underset{\sim}{\theta}_1 - \underset{\sim}{\lambda}_1)' \underset{\sim}{V}_1^{-1} \underset{\sim}{Q}^{-1} \underset{\sim}{V}_2^{-1} (\underset{\sim}{\theta}_2 - \underset{\sim}{\lambda}_2) ,$$

as $c \to 0$.

Next we prove the uniform integrability in $c \leq c_o$ of the left hand side of (4.41) for case (ii). However, by (4.34) and (4.35), we need only bound,

$$E_{\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2} \left[c^{-2} \frac{\ddot{s}_N^4}{N^2 \|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}_1\|_{V_1}^2} \frac{\bar{s}_M^{-4}}{M^2 \|\bar{\underset{\sim}{Y}}_M - \underset{\sim}{\lambda}_2\|_{V_2}^2} \right] \qquad (4.42)$$

uniformly in $c \leq c_o$ to prove the uniform integrability of the left hand side of (4.40). With this end note that

$$(4.42) = \sum_{n=n_o}^{\infty} \sum_{m=n_o}^{\infty} E_{\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2} \left[\frac{\ddot{s}_n^4}{cn^2 \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}_1\|_{V_1}^2} \frac{\bar{s}_m^{-4}}{cm^2 \|\bar{\underset{\sim}{Y}}_m - \underset{\sim}{\lambda}_2\|_{V_2}^2} \right] \qquad (4.43)$$

$$\cdot I_{[N=n, M=m]} \Big]$$

$$= \Sigma_{n=n_o}^{\infty}\Sigma_{m=n_o}^{\infty} E_{\sigma_1^2,\sigma_2^2}\left[\frac{\ddot{s}_n^4}{cn\sigma_1^2}\frac{\bar{s}_m^4}{cm\sigma_2^2} I_{[N=n,M=m]}\right] g_p(\Delta_n)g_p(\Delta_m)$$

$$= \frac{1}{\sigma_1^2}\frac{1}{\sigma_2^2} E_{\sigma_1^2,\sigma_2^2}\left[g_p(\Delta_N)\frac{\ddot{s}_N^4}{cN} g_p(\Delta_M)\frac{\bar{s}_M^4}{cM}\right] .$$

Following the proof of Theorem 2.1 we have with probability one, $g_p(\Delta_N)\frac{\ddot{s}_N^4}{cN} \le L_1\ddot{s}_N^4/(Nc^{1/2})^2$ and $g_p(\Delta_M)\frac{\bar{s}_M^4}{cM} \le L_2\bar{s}_M^4/(Mc^{1/2})^2$ where L_1 and L_2 are constants not depending on c. Thus it suffices to bound, uniformly in $c \le c_o$,

$$E_{\sigma_1^2,\sigma_2^2}\left[\frac{\ddot{s}_N^4}{(Nc^{1/2})^2}\frac{\bar{s}_M^4}{(Mc^{1/2})^2}\right] . \tag{4.44}$$

With this end, using Holder's inequality, we have

$$(4.44) \le E_{\sigma_1^2}^{1/2}\left[\frac{\ddot{s}_N^8}{(Nc^{1/2})^4}\right] E_{\sigma_2^2}^{1/2}\left[\frac{\bar{s}_M^8}{(Mc^{1/2})^4}\right] . \tag{4.45}$$

Using Holder's inequality once more with $\delta \in (0,1)$,

$$E_{\sigma_1^2}\left[\frac{\ddot{s}_N^8}{(Nc^{1/2})^4}\right] \le E_{\sigma_1^2}\left[(N^2c)^{-2\frac{2+\delta}{2}}\right]^{\frac{2}{2+\delta}} E_{\sigma_1^2}\left[(\ddot{s}_N^8)^{\frac{2+\delta}{\delta}}\right]^{\frac{\delta}{2+\delta}} . \tag{4.46}$$

By the backward martingale property of $\ddot{s}_n^2$ we can bound

$E_{\sigma_1^2}[(\bar{s}_N^{-8})^{\frac{2+\delta}{\delta}}]$ by a constant independent of c. Additionally, for ε in (0,1),

$$E_{\sigma_1^2}[(N^2c)^{-(2+\delta)}] \tag{4.47}$$

$$\leq E_{\sigma_1^2}[(N^2c)^{-(2+\delta)} I_{[N\leq \varepsilon n^*]} + (\varepsilon^2 n^{*2} c)^{-(2+\delta)}]$$

$$\leq (n_o^2 c)^{-(2+\delta)} P_{\sigma_1^2}(N \leq \varepsilon n^*) + (\varepsilon^2 \sigma_1^2 \operatorname{tr}(\underset{\sim}{Q}\underset{\sim}{V}_1))^{-(2+\delta)} .$$

Using the result that $P_{\sigma_1^2}(N \leq \varepsilon n^*) = O(c^{(\frac{1}{2})(n_o-1)p})$, then for $n_o \geq 3$ and $c \leq c_o < 1$,

$$E_{\sigma_1^2}[(N^2c)^{-(2+\delta)}] \tag{4.48}$$

$$\leq K_o (n_o^2)^{-(2+\delta)} c_o^{\frac{1}{2}(n_o-1)p-(2+\delta)} + (\varepsilon^2 \sigma_1^2 \operatorname{tr}(\underset{\sim}{Q}\underset{\sim}{V}_1))^{-(2+\delta)} ,$$

where K_o is a constant independent of c. Similarly, $E_{\sigma_1^2}\left[\frac{\bar{s}_M^{-8}}{(Mc^{1/2})^4}\right]$ can be bounded by a constant independent of c. Combining (4.42) - (4.48) for $n_o \geq 3$, the left hand side of (4.41) is uniformly integrable in $c \leq c_o$. Consequently, for $n_o \geq 3$, in case (ii), as $c \to 0$,

$$c^{-1}E_{\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2}\Big[\frac{\overset{\cdot\cdot}{s}_N^2}{N\|\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}_1\|_{\underset{\sim}{V}_1}^2}\frac{\overset{-}{s}_M^2}{M\|\bar{\underset{\sim}{Y}}_M-\underset{\sim}{\lambda}_2\|_{\underset{\sim}{V}_2}^2}(\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}_1)'\underset{\sim}{V}_1^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}_2^{-1}(\bar{\underset{\sim}{Y}}_M-\underset{\sim}{\lambda}_2) \quad (4.49)$$

$$+\left(\frac{p}{p+2}\right)^2\sigma_1\|\underset{\sim}{\theta}_1-\underset{\sim}{\lambda}_1\|_{\underset{\sim}{V}_1}^{-2}\sigma_2\|\underset{\sim}{\theta}_2-\underset{\sim}{\lambda}_2\|_{\underset{\sim}{V}_2}^{-2}(\mathrm{tr}(\underset{\sim}{QV}_1)\mathrm{tr}(\underset{\sim}{QV}_2))^{-1/2}$$

$$\cdot(\underset{\sim}{\theta}_1-\underset{\sim}{\lambda}_1)'\underset{\sim}{V}_1^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}_2^{-1}(\underset{\sim}{\theta}_2-\underset{\sim}{\lambda}_2)\ .$$

Hence, using (4.27) - (4.28) and (4.49) we have for $n_0 \geq 3$, case (ii), as $c \to 0$,

$$R(\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2,\underset{\sim}{\delta}_{N,M}^{b_1,b_2}) \quad (4.50)$$

$$= 2c^{1/2}[(\sigma_1^2\,\mathrm{tr}(\underset{\sim}{QV}_1))^{1/2}+(\sigma_2^2\,\mathrm{tr}(\underset{\sim}{QV}_2))^{1/2}]+\frac{c}{p}$$

$$-cb_1\left(\frac{p}{p+2}\right)^2(2(p^2-4)p^{-1}-b_1)\sigma_1^2(\mathrm{tr}\ \underset{\sim}{QV}_1)^{-1}\|\underset{\sim}{\theta}_1-\underset{\sim}{\lambda}_1\|_{\underset{\sim}{V}_1}^{-2}$$

$$-cb_2\left(\frac{p}{p+2}\right)^2(2(p^2-4)p^{-1}-b_2)\sigma_2^2(\mathrm{tr}\ \underset{\sim}{QV}_2)^{-1}\|\underset{\sim}{\theta}_2-\underset{\sim}{\lambda}_2\|_{\underset{\sim}{V}_2}^{-2}$$

$$-2cb_1b_2\left(\frac{p}{p+2}\right)^2\sigma_1\|\underset{\sim}{\theta}_1-\underset{\sim}{\lambda}_1\|_{\underset{\sim}{V}_1}^{-2}\sigma_2\|\underset{\sim}{\theta}_2-\underset{\sim}{\lambda}_2\|_{\underset{\sim}{V}_2}^{-2}(\mathrm{tr}(\underset{\sim}{QV}_1)\mathrm{tr}(\underset{\sim}{QV}_2))^{-1/2}$$

$$\cdot(\underset{\sim}{\theta}_1-\underset{\sim}{\lambda}_1)'\underset{\sim}{V}_1^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}_2^{-1}(\underset{\sim}{\theta}_2-\underset{\sim}{\lambda}_2)+o(c)\ .$$

Thus, in this case

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{\underset{\sim}{W}}_{N,M}) - R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \hat{\underset{\sim}{\delta}}_{N,M}^{b_1,b_2}) \tag{4.51}$$

$$= c\left(\frac{p}{p+2}\right)^2 [b_1(2\,\frac{p^2-4}{p} - b_1)\ \sigma_1^2 (\text{tr}\ \underset{\sim}{QV}_1)^{-1}\ \|\underset{\sim}{\theta}_1 - \underset{\sim}{\lambda}_1\|_{V_1}^{-2}$$

$$+ b_2(2\,\frac{p^2-4}{p} - b_2)\ \sigma_2^2 (\text{tr}\ \underset{\sim}{QV}_2)^{-1}\ \|\underset{\sim}{\theta}_2 - \underset{\sim}{\lambda}_2\|_{V_2}^{-2}$$

$$+ 2b_1 b_2\ \sigma_1 \|\underset{\sim}{\theta}_1 - \underset{\sim}{\lambda}_1\|_{V_1}^{-2} \sigma_2 \|\underset{\sim}{\theta}_2 - \underset{\sim}{\lambda}_2\|_{V_2}^{-2} \{\text{tr}(\underset{\sim}{QV}_1)\text{tr}(\underset{\sim}{QV}_2)\}^{-1/2}$$

$$\cdot\ (\underset{\sim}{\theta}_1 - \underset{\sim}{\lambda}_1)' \underset{\sim}{V}_1^{-1} \underset{\sim}{Q}^{-1} \underset{\sim}{V}_2^{-1} (\underset{\sim}{\theta}_2 - \underset{\sim}{\lambda}_2)] + o(c)\ .$$

For case (iii) $\underset{\sim}{\theta}_1 = \underset{\sim}{\lambda}_1$, $\underset{\sim}{\theta}_2 \neq \underset{\sim}{\lambda}_2$, using Anscombe's Theorem, as $c \to 0$,

$$\underset{\sim}{B}_{N1} \xrightarrow{L} N(\underset{\sim}{0}, \sigma_1^2 \underset{\sim}{V}_1) \tag{4.52}$$

and hence

$$-\underset{\sim}{B}_{N1} \xrightarrow{L} N(\underset{\sim}{0}, \sigma_1^2 \underset{\sim}{V}_1) \tag{4.53}$$

Consequently, using (4.52) - (4.53)

$$\frac{\underset{\sim}{B}_{N1}}{\|\underset{\sim}{B}_{N1}\|_{V_1}^2} \xrightarrow{L} \underset{\sim}{Z} \text{ (say)} \tag{4.54}$$

and

$$\frac{-B_{N1}}{\|B_{N1}\|_{V_1}^2} \xrightarrow{L} Z . \tag{4.55}$$

Here (4.54) and (4.55) imply that $E[Z] = 0$. Consequently, using previous results and (4.54), as $c \to 0$,

$$c^{-3/4} \frac{\ddot{s}_N^2}{N\|\bar{X}_N - \lambda_1\|_{V_1}^2} \frac{\bar{s}_M^2}{M\|\bar{Y}_M - \lambda_2\|_{V_2}^2} (\bar{X}_N - \lambda_1)' V_1^{-1} Q^{-1} V_2^{-1} (\bar{Y}_M - \lambda_2) \tag{4.56}$$

$$\xrightarrow{L} \left(\frac{p}{p+2}\right)^2 \frac{\sigma_1^2}{(\sigma_1^2 \operatorname{tr}(QV_1))^{1/4}} \frac{\sigma_2^2}{(\sigma_2^2 \operatorname{tr}(QV_2))^{1/2}} Z' V_1^{-1} Q^{-1} V_2^{-1} (\theta_2 - \lambda_2) .$$

Next we prove the uniform integrability in $c \le c_0$ of the left hand side of (4.56) for case (iii). By (4.34) and (4.35) to prove the uniform integrability in $c \le c_0$ of the left hand side of (4.56) we need only bound,

$$E_{\theta_1, \theta_2, \sigma_1^2, \sigma_2^2}\left[c^{-3/2} \frac{\ddot{s}_N^4}{N^2\|\bar{X}_N - \lambda_1\|_{V_1}^2} \frac{\bar{s}_M^4}{M^2\|\bar{Y}_M - \lambda_2\|_{V_2}^2}\right] \tag{4.57}$$

uniformly in $c \le c_0$. Using (4.36) and (4.43) we have

$$(4.57) = \frac{1}{\sigma_1^2}\frac{1}{\sigma_2^2} E_{\sigma_1^2, \sigma_2^2}\left[K_1 \frac{\ddot{s}_N^4}{Nc^{1/2}} g_p(A_M) \frac{\bar{s}_M^4}{Mc}\right]. \tag{4.58}$$

By previous arguments we have

$$\text{rhs of (4.58)} \le \frac{K_1}{\sigma_1^2}\frac{L_1}{\sigma_2^2} E_{\sigma_1^2,\sigma_2^2}\left[\frac{\ddot{s}_N^{-4}}{Nc^{1/2}} \frac{\bar{s}_M^{-4}}{M^2 c}\right] . \tag{4.59}$$

Following previous arguments, for $c \le c_0 < 1$ and $n_0 \ge 3$, rhs of (4.59) can be bounded by a constant independent of c. Combining (4.57) - (4.59) the uniform integrability of the left hand side of (4.56) follows. Hence, for $n_0 \ge 3$, case (iii), as $c \to 0$,

$$c^{-3/2} E_{\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2}\left[\frac{\ddot{s}_N^{-2}}{N\|\bar{X}_N-\lambda_1\|^2_{V_1}} \frac{\bar{s}_M^{-2}}{M\|\bar{Y}_M-\lambda_2\|^2_{V_2}} (\bar{X}_N-\underset{\sim}{\lambda}_1)' V_1^{-1} Q^{-1} V_2^{-1} (\bar{Y}_M-\underset{\sim}{\lambda}_2)\right] \tag{4.60}$$

$$+ \left(\frac{p}{p+2}\right)^2 \frac{\sigma_1^2}{(\sigma_1^2 \operatorname{tr}(QV_1))^{1/4}} \frac{\sigma_2^2}{(\sigma_2^2 \operatorname{tr}(QV_2))^{1/2}} E[Z' V_1^{-1} Q^{-1} V_2^{-1} (\underset{\sim}{\theta}_2-\underset{\sim}{\lambda}_2)$$

$$= 0.$$

Consequently, using (4.25), (4.28) and (4.60), for $n_0 \ge 3$, case (iii), as $c \to 0$,

$$R(\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2,\delta_{N,M}^{b_1,b_2}) \tag{4.61}$$

$$= 2c^{1/2}\left[(\sigma_1^2 \operatorname{tr}(QV_1))^{1/2} + (\sigma_2^2 \operatorname{tr}(QV_2))^{1/2}\right]$$

$$- c^{1/2} \sigma_1 b_1 \left(\frac{p}{p+2}\right)^2 (2(p^2-4)p^{-1} - b_1)(\operatorname{tr} \underset{\sim}{QV}_1)^{-1/2} K_1 + o(c^{1/2}) .$$

Thus, in this case,

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{\underset{\sim}{W}}_{N,M}) - R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \underset{\sim}{\delta}^{b_1,b_2}_{N,M}) \tag{4.62}$$

$$= c^{1/2} \sigma_1 \left(\frac{p}{p+2}\right)^2 b_1 \left(2\,\frac{p^2-4}{p} - b_1\right)(\operatorname{tr} \underset{\sim}{QV}_1)^{-1/2} K_1 + o(c^{1/2}) .$$

For case (iv) $\underset{\sim}{\theta}_1 \neq \underset{\sim}{\lambda}_1$, $\underset{\sim}{\theta}_2 = \underset{\sim}{\lambda}_2$, following the arguments of case (iii), we have for $n_0 \geq 3$, as $c \to 0$,

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \underset{\sim}{\delta}^{b_1,b_2}_{N,M}) \tag{4.63}$$

$$= 2c^{1/2} [(\sigma_1^2 \operatorname{tr}(\underset{\sim}{QV}_1))^{1/2} + (\sigma_2^2 \operatorname{tr}(\underset{\sim}{QV}_2))^{1/2}]$$

$$- c^{1/2} \sigma_2 b_2 \left(\frac{p}{p+2}\right)^2 (2(p^2-4)p^{-1} - b_2)(\operatorname{tr} \underset{\sim}{QV}_2)^{-1/2} K_2 + o(c^{1/2}) .$$

Thus, in this case

$$R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \bar{\underset{\sim}{W}}_{N,M}) - R(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \sigma_1^2, \sigma_2^2, \underset{\sim}{\delta}^{b_1,b_2}_{N,M}) \tag{4.64}$$

$$= c^{1/2} \sigma_2 \left(\frac{p}{p+2}\right)^2 b_2 (2(p^2-4)p^{-1} - b_2)(\operatorname{tr} \underset{\sim}{QV}_2)^{-1/2} K_2 + o(c^{1/2}) .$$

Remarks

For case (i), the leading term in the risk difference $R(\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2,\underset{\sim}{\bar{W}}_{N,M}) - R(\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2,\underset{\sim}{\delta}^{b_1,b_2}_{N,M})$ is maximized at $b_1 = b_2 = p-2$ when we restrict ourselves to the intervals $(0,p-2]$ for values of b_1 and b_2. Accordingly, the asymptotic risk expansion suggests the optimal $b_1 = b_2 = p-2$. This is different from what we have seen in Chapter 2, since there we allowed the interval $(0,2(p-2))$ for b, and accordingly, the optimal b turned out to be $(p^2-4)/p$. For case (iii), the optimal choice of b_1 in the interval $(0,p-2]$ turns out to be $b_1 = p-2$, although we cannot say anything about the optimal choice of b_2. Case (iv) is just the opposite where the optimal b_2 (in the interval $(0,p-2]$) turns out to be $p-2$, but nothing can be said about the optimal b_1. The optimal choice of b_1 and b_2 in case (ii) seems to be the most unclear, since the leading term in the risk difference $R(\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2,\underset{\sim}{\bar{W}}_{N,M}) - R(\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\sigma_1^2,\sigma_2^2,\underset{\sim}{\delta}^{b_1,b_2}_{N,M})$ does not reveal any optimal b_1 and b_2 free from the unknown parameters.

CHAPTER FIVE
SEQUENTIAL SHRINKAGE ESTIMATION OF INDEPENDENT NORMAL MEANS WITH UNKNOWN VARIANCES

5.1 Introduction

Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ be a sequence of independent and identically distributed $N(\underset{\sim}{\theta},\underset{\sim}{\Sigma})$ variables where $\underset{\sim}{\theta} \in R^p$ is unknown and $\underset{\sim}{\Sigma} = \text{diag}(\sigma_1^2,\ldots,\sigma_p^2)$. The problem is estimation of $\underset{\sim}{\theta}$. Based on $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$, if $\underset{\sim}{\delta}_n = \underset{\sim}{\delta}_n(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n)$ is used to estimate $\underset{\sim}{\theta}$, suppose the loss incurred is given by

$$L(\underset{\sim}{\theta},\underset{\sim}{\delta}_n) = (\underset{\sim}{\delta}_n - \underset{\sim}{\theta})' \underset{\sim}{Q} (\underset{\sim}{\delta}_n - \underset{\sim}{\theta}) + cn, \tag{5.1}$$

where $\underset{\sim}{Q} = \text{diag}(q_1,\ldots,q_p)$ is known with $q_j > 0$, $j=1,\ldots,p$ and $c(>0)$ is the known cost per sampling unit. Stated differently, the problem could be thought of as simultaneously estimating p independent normal means with unequal variances, where we sample an equal number of times from each population. For a fixed sample of size n, the most natural estimator of $\underset{\sim}{\theta}$ is the sample mean vector $\bar{\underset{\sim}{X}}_n = \frac{1}{n} \Sigma_{i=1}^n \underset{\sim}{X}_i$, with risk

$$R(\underset{\sim}{\theta},\underset{\sim}{\Sigma},\bar{\underset{\sim}{X}}_n) = \frac{1}{n} \text{tr}(\underset{\sim\sim}{Q\Sigma}) + cn \ . \tag{5.2}$$

If $\underset{\sim}{\Sigma}$ is known the above risk is minimized at $n = n^* \equiv (\text{tr } \underset{\sim\sim}{Q\Sigma}/c)^{1/2}$. Note here that $\text{tr}(\underset{\sim\sim}{Q\Sigma}) = \Sigma_{j=1}^{p} q_j \sigma_j^2$. However, if $\underset{\sim}{\Sigma}$ is unknown, there does not exist any fixed sample size which minimizes (5.2) simultaneously for all $\sigma_1^2, \ldots, \sigma_p^2$. Motivated by the definition of n^* we proposed the following sequential sampling scheme.

$$N = \inf\{n \geq m : n \geq (\text{tr } \underset{\sim\sim}{QW}_n/c)^{1/2}\} , \tag{5.3}$$

where $m(\geq 2)$ denotes the initial sample size, $\underset{\sim}{W}_n = \text{diag}(s_{n1}^2, \ldots, s_{np}^2)$ and for $j=1,\ldots,p$,

$$s_{nj}^2 = \frac{1}{n-1} \Sigma_{k=1}^{n} (X_{kj} - \bar{X}_{nj})^2 , \tag{5.4}$$

for every $n \geq 2$ and $\bar{X}_{nj} = \frac{1}{n} \Sigma_{k=1}^{n} X_{kj}$. The above stopping rule was first proposed by Ghosh et al. (1976) and Woodroofe (1977) for the more general case of an unknown positive definite $\underset{\sim}{\Sigma}$.

To show that the above rule (5.3) can be motivated from a minimax criterion we need only follow the arguments of Section 1.1 with $\sigma^2 \underset{\sim}{V}$ replaced by $\underset{\sim}{\Sigma}$ and $\sigma^2 \underset{\sim}{B}$, the prior covariance matrix of $\underset{\sim}{\theta}$, replaced by $\underset{\sim}{Z} = \text{diag}(z_1, \ldots, z_p)$. Then the Bayes estimator of $\underset{\sim}{\theta}$ is

$$\underset{\sim}{\delta}_n^B(\underset{\sim}{\bar{X}}_n) = \underset{\sim}{\mu} + (\underset{\sim}{I}_p + \frac{1}{n} \underset{\sim\sim}{\Sigma Z}^{-1})^{-1} (\underset{\sim}{\bar{X}}_n - \underset{\sim}{\mu}) ,$$

with corresponding posterior risk equal to $\frac{1}{n} \text{tr}[\underset{\sim\sim}{Q\Sigma}(\underset{\sim}{I}_p + \frac{1}{n} \underset{\sim}{Z}^{-1} \underset{\sim}{\Sigma})^{-1}] + cn$. Again, the posterior risk is independent of any unknowns so that the

Bayes sequential decision rule is a fixed sample rule with sample size determined by minimizing $\frac{1}{n}\,\mathrm{tr}[\underset{\sim}{Q}\underset{\sim}{\Sigma}(\underset{\sim}{I}_p + \frac{1}{n}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\Sigma})^{-1}] + cn$ with respect to n.

Next consider the sequence of $\{N(\underset{\sim}{\mu}, m\underset{\sim}{I}_p),\ m \geq 1\}$ priors on $\underset{\sim}{\Theta}$. Then the corresponding sequence of Bayes estimators is

$$\underset{\sim}{\delta}^B_n(\bar{\underset{\sim}{X}}_n) = \underset{\sim}{\mu} + (\underset{\sim}{I}_p + \frac{1}{nm}\underset{\sim}{\Sigma})^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\mu})$$

with the sequence of risks being $\frac{1}{n}\,\mathrm{tr}[\underset{\sim}{Q}\underset{\sim}{\Sigma}(\underset{\sim}{I}_p + \frac{1}{nm}\underset{\sim}{\Sigma})^{-1}] + cn$. Since $\frac{1}{n}\,\mathrm{tr}[\underset{\sim}{Q}\underset{\sim}{\Sigma}(\underset{\sim}{I}_p + \frac{1}{nm}\underset{\sim}{\Sigma})^{-1}] + cn \to \frac{1}{n}\,\mathrm{tr}(\underset{\sim}{Q}\underset{\sim}{\Sigma}) + cn = K$ as $m \to \infty$ and $R(\underset{\sim}{\theta}, \underset{\sim}{\Sigma}, \bar{\underset{\sim}{X}}_n) = K$ for all $\underset{\sim}{\theta} \in R^p$ then n*, the sample size, and $\bar{\underset{\sim}{X}}_n$, the estimator, is a sequential minimax rule. Consequently, we can consider (5.3) as an empirical minimax stopping rule.

In Section 5.2 we develop an asymptotic risk expansion up to the second order term for $\bar{\underset{\sim}{X}}_N$ and for a class of rivial estimators $\underset{\sim}{\delta}^B_N(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N)$, where

$$\underset{\sim}{\delta}^b_n(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n) = \bar{\underset{\sim}{X}}_n - \frac{b}{n\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|^2_W}\,\underset{\sim}{Q}^{-1}\underset{\sim}{W}_n^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}) \tag{5.5}$$

for every $n \geq 2$, $\|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|^2_W = (\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda})'\underset{\sim}{W}_n^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{W}_n^{-1}(\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda})$, b is a constant and $\underset{\sim}{\lambda} \in R^p$ is the known point towards which we want to shrink.

5.2 Asymptotic Risk Expansion for $\bar{X}_N$ and δ^b_N

First we obtain an asymptotic (as $c \to 0$) risk expansion for the risk of $\bar{X}_N$ and then for δ^b_N. Observe that

$$R(\underset{\sim}{\theta}, \underset{\sim}{\Sigma}, \bar{X}_N) = E_{\underset{\sim}{\theta}, \underset{\sim}{\Sigma}}[(\bar{X}_N - \underset{\sim}{\theta})' Q (\bar{X}_N - \underset{\sim}{\theta})] + cE_{\underset{\sim}{\Sigma}}[N] \tag{5.6}$$

$$= tr(Q\Sigma) E_{\underset{\sim}{\Sigma}}[N^{-1}] + cE_{\underset{\sim}{\Sigma}}[N]$$

$$= 2c^{1/2} (tr\ \Sigma Q)^{1/2} + cE_{\underset{\sim}{\Sigma}}[\frac{(N-n^*)^2}{N}] .$$

From definition $N \xrightarrow{a.s.} \infty$ as $c \to 0$ so that $\Sigma_{j=1}^p q_j s_{nj}^2 \xrightarrow{a.s.} \Sigma_{j=1}^p q_j \sigma_j^2$. Also, using the Helmert orthogonal transformation $Y_{kj} = (X_{1j} + \ldots + X_{k-1,j} - (k-1)X_{kj})/\sqrt{k(k-1)}$, $j=1,\ldots,p$ and $k=2,\ldots,n$, write

$$tr(Q W_n) = \Sigma_{j=1}^p q_j s_{nj}^2 \tag{5.7}$$

$$= \frac{1}{n-1} \Sigma_{j=1}^p q_j [\Sigma_{k=2}^n Y_{kj}^2]$$

$$= \frac{1}{n-1} \Sigma_{k=2}^n [\Sigma_{j=1}^p q_j Y_{kj}^2] .$$

Hence $tr(Q W_n)$ is the mean of $(n-1)$ independent and identically distributed random variables with mean $tr(Q\Sigma) = \Sigma_{j=1}^p q_j \sigma_j^2$ and

variance $2\mathrm{tr}(\underset{\sim}{Q}^2\underset{\sim}{\Sigma}^2) = 2\sum_{j=1}^{p} q_j^2\sigma_j^4$. Therefore, using Anscombe's Theorem $\sqrt{N}(\mathrm{tr}(\underset{\sim}{QW}_N) - \mathrm{tr}(\underset{\sim}{Q\Sigma})) \xrightarrow{L} N(0, 2\mathrm{tr}(\underset{\sim}{Q}^2\underset{\sim}{\Sigma}^2))$ so that using the delta method, $\sqrt{N}((\mathrm{tr}\ \underset{\sim}{QW}_N)^{1/2} - (\mathrm{tr}\ \underset{\sim}{Q\Sigma})^{1/2}) \xrightarrow{L} N(0, 1/2\,(\mathrm{tr}\ \underset{\sim}{Q}^2\underset{\sim}{\Sigma}^2)/(\mathrm{tr}\ \underset{\sim}{Q\Sigma}))$.

Next use the inequality

$$(\mathrm{tr}\ \underset{\sim}{QW}_N/c)^{1/2} \le N \le m + (\mathrm{tr}\ \underset{\sim}{QW}_{N-1}/c)^{1/2}. \tag{5.8}$$

Dividing all sides of (5.8) by n^*, making $c \to 0$, and using $(\mathrm{tr}\ \underset{\sim}{QW}_N)^{1/2} \xrightarrow{a.s.} (\mathrm{tr}\ \underset{\sim}{Q\Sigma})^{1/2}$, it follows that $N/n^* \xrightarrow{a.s.} 1$ as $c \to 0$. Thus $\sqrt{n^*}((\mathrm{tr}\ \underset{\sim}{QW}_N)^{1/2} - (\mathrm{tr}\ \underset{\sim}{Q\Sigma})^{1/2}) \xrightarrow{L} N(0, 1/2\,(\mathrm{tr}\ \underset{\sim}{Q}^2\underset{\sim}{\Sigma}^2)/(\mathrm{tr}\ \underset{\sim}{Q\Sigma}))$.

Again from (5.8),

$$\sqrt{n^*}\,((\mathrm{tr}\ \underset{\sim}{QW}_N)^{1/2} - (\mathrm{tr}\ \underset{\sim}{Q\Sigma})^{1/2})/(\mathrm{tr}\ \underset{\sim}{Q\Sigma})^{1/2} \tag{5.9}$$

$$\le (N-n^*)/\sqrt{n^*} \le m/\sqrt{n^*} + \sqrt{n^*}\,((\mathrm{tr}\ \underset{\sim}{QW}_{N-1})^{1/2} - (\mathrm{tr}\ \underset{\sim}{Q\Sigma})^{1/2})/(\mathrm{tr}\ \underset{\sim}{Q\Sigma})^{1/2}.$$

Hence from (5.9), as $c \to 0$, $(N-n^*)/\sqrt{n^*} \xrightarrow{L} N(0, 1/2\,(\mathrm{tr}\ \underset{\sim}{Q}^2\underset{\sim}{\Sigma}^2)/(\mathrm{tr}\ \underset{\sim}{Q\Sigma})^2)$, implying that $(N-n^*)/\sqrt{N} \xrightarrow{L} N(0, 1/2\,(\mathrm{tr}\ \underset{\sim}{Q}^2\underset{\sim}{\Sigma}^2)/(\mathrm{tr}\ \underset{\sim}{Q\Sigma})^2)$ as $c \to 0$ since $N/n^* \xrightarrow{a.s.} 1$ as $c \to 0$. Also, the uniform integrability of $(N-n^*)^2/N$ for all $c \le c_o$ (specified) can be proved by repeating essentially the arguments of Ghosh and Mukhopadhyay (1980). Thus $E_{\underset{\sim}{\Sigma}}[(N-n^*)^2/N] \to (1/2\,(\mathrm{tr}\ \underset{\sim}{Q}^2\underset{\sim}{\Sigma}^2)/(\mathrm{tr}\ \underset{\sim}{Q\Sigma})^2)$ as $c \to 0$, and thus, from (5.6), as $c \to 0$,

$$R(\underset{\sim}{\theta}, \underset{\sim}{\Sigma}, \bar{\underset{\sim}{X}}_N) = 2c^{1/2} (\text{tr } Q\underset{\sim}{\Sigma})^{1/2} + \frac{c}{2} (\text{tr } Q^2\underset{\sim}{\Sigma}^2)/(\text{tr } Q\underset{\sim}{\Sigma})^2 + o(c) . \quad (5.10)$$

Next we develop the asymptotic risk expansion of δ^b_N. First note that

$$R(\underset{\sim}{\theta}, \underset{\sim}{\Sigma}, \delta^b_N) = R(\underset{\sim}{\theta}, \underset{\sim}{\Sigma}, \bar{\underset{\sim}{X}}_N) \quad (5.11)$$

$$- 2bE_{\underset{\sim}{\theta}, \underset{\sim}{\Sigma}}\left[\frac{1}{N \|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|^2_W} (\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda})' W_N^{-1} (\bar{\underset{\sim}{X}}_N - \underset{\sim}{\theta})\right]$$

$$+ b^2 E_{\underset{\sim}{\theta}, \underset{\sim}{\Sigma}}\left[\frac{1}{N^2 \|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|^2_W}\right] .$$

Denote by B_{nj} the σ-algebra generated by Y_{ij}, $i=2,\ldots,n$, $j=1,\ldots,p$, and $\bar{X}_{ni}$, $i\neq j=1,\ldots,p$. Then

$$E_{\underset{\sim}{\theta}, \underset{\sim}{\Sigma}}\left[\frac{1}{N \|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|^2_W} (\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda})' W_N^{-1} (\bar{\underset{\sim}{X}}_N - \underset{\sim}{\theta})\right] \quad (5.12)$$

$$= \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\theta}, \underset{\sim}{\Sigma}}\left[\frac{1}{n \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|^2_W} (\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda})' W_n^{-1} (\bar{\underset{\sim}{X}}_n - \underset{\sim}{\theta}) I_{[N=n]}\right]$$

$$= \Sigma_{n=m}^{\infty} \Sigma_{j=1}^{p} E_{\underset{\sim}{\theta}, \underset{\sim}{\Sigma}}\left[\frac{1}{n \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|^2_W} (\bar{X}_{nj} - \lambda_j)\, s_{nj}^{-2} (\bar{X}_{nj} - \theta_j) I_{[N=n]}\right]$$

$$= \Sigma_{n=m}^{\infty} \Sigma_{j=1}^{p} E_{\underset{\sim}{\theta}, \underset{\sim}{\Sigma}} E_{\underset{\sim}{\theta}, \underset{\sim}{\Sigma}}\left[\frac{1}{n \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|^2_W} (\bar{X}_{nj} - \lambda_j)\, s_{nj}^{-2} (\bar{X}_{nj} - \theta_j) I_{[N=n]} | B_{nj}\right]$$

$$= \Sigma_{n=m}^{\infty} \Sigma_{j=1}^{p} E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} [I_{[N=n]} E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} \{ \frac{1}{n \| \bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda} \|_W^2} (\bar{X}_{nj} - \lambda_j) s_{nj}^{-2} (\bar{X}_{nj} - \Theta_j) | B_{nj} \}] ,$$

where in the last equality in (5.12) we use the fact that $I_{[N=n]}$ is B_{nj} measurable j=1,...,p. By Stein's identity and the mutual independence $\bar{X}_{n1}$, . . . , $\bar{X}_{np}$ and $\underset{\sim}{W}_n$, for j=1,...,p,

$$E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} [\frac{1}{n \| \bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda} \|_W^2} (\bar{X}_{nj} - \lambda_j) \; s_{nj}^{-2} (\bar{X}_{nj} - \Theta_j) | B_{nj}] \tag{5.13}$$

$$= E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} [\frac{\sigma_j^2}{n} \frac{\partial}{\partial \bar{X}_{nj}} \frac{(\bar{X}_{nj} - \lambda_j) s_{nj}^{-2}}{n \| \bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda} \|_W^2} | B_{nj}]$$

$$= \frac{\sigma_j^2}{n} E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} [(\frac{s_{nj}^{-2}}{n \| \bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda} \|_W^2} - 2 \frac{s_{nj}^{-2}}{n \| \bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda} \|_W^2} \frac{(\bar{X}_{nj} - \lambda_j)^2}{q_j s_{nj}^4}) | B_{nj}] .$$

Using (5.12) and (5.13),

$$R(\underset{\sim}{\Theta}, \underset{\sim}{\Sigma}, \delta_N^b) = R(\underset{\sim}{\Theta}, \underset{\sim}{\Sigma}, \bar{\underset{\sim}{X}}_N) - 2bE_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} [\frac{1}{N^2 \| \bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda} \|_W^2} \Sigma_{j=1}^{p} \frac{\sigma_j^2}{s_{Nj}^2}] \tag{5.14}$$

$$+ 4bE_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} [\frac{1}{N^2 \| \bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda} \|_W^4} \Sigma_{j=1}^{p} \frac{(\bar{X}_{Nj} - \lambda_j)^2}{q_j s_{Nj}^4} \frac{\sigma_j^2}{s_{Nj}^2}]$$

$$+ b^2 E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} [\frac{1}{N^2 \| \bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda} \|_W^2}] .$$

For technical reasons the remainder of the asymptotic risk expansion will be broken up into the two cases $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$ and $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$.

For $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$ denote by B_n the σ-algebra generated by Y_{ij}, $i=2,\ldots,n$, $j=1,\ldots,p$. Then realizing that $I_{[N=n]}$ is B_n measurable,

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2 \|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|_W^2}\right] \tag{5.15}$$

$$= \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{n^2 \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|_W^2} I_{[N=n]}\right]$$

$$= \Sigma_{n=m}^{\infty} \frac{1}{n} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{n \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|_W^2} I_{[N=n]} \mid B_n\right] .$$

$$= \Sigma_{n=m}^{\infty} \frac{1}{n} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[I_{[N=n]} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{n \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|_W^2} \mid B_n\right]\right] .$$

From Cressie et al. (1981) and the independence of $\bar{\underset{\sim}{X}}_n$ and Y_{ij}, $i=2,\ldots,n$, $j=1,\ldots,p$, we have

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{n \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|_W^2} \mid B_n\right] \tag{5.16}$$

$$= \int_0^{\infty} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\exp(-tn \|\bar{\underset{\sim}{X}}_n - \underset{\sim}{\lambda}\|_W^2) \mid B_n\right] dt$$

$$= \int_0^{\infty} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\exp\left(-tn \sum_{j=1}^{p} \frac{(\bar{X}_{nj} - \lambda_j)^2}{q_j s_{nj}^4}\right) \mid B_n\right] dt$$

$$= \int_0^\infty \prod_{j=1}^{p} E_{\underset{\sim}{\Theta}, \underset{\sim}{\Sigma}}[\exp\left(-tn \frac{(\bar{X}_{nj} - \lambda_j)^2}{q_j s_{nj}^4}\right) \mid B_n]dt ,$$

where in the last equality in (5.16) we use the mutual independence of the $\bar{X}_{nj}$'s, j=1,...,p. Using again the mgf's of chisquared random variables for each j=1,...,p,

$$E_{\underset{\sim}{\Theta}, \underset{\sim}{\Sigma}}[\exp(-tn \frac{(\bar{X}_{nj} - \lambda_j)^2}{q_j s_{nj}^4}) \mid B_n] \tag{5.17}$$

$$= \left(\frac{q_j s_{nj}^4}{q_j s_{nj}^4 + 2t\sigma_j^2}\right)^{1/2} \exp(-tn \frac{(\Theta_j - \lambda_j)^2}{q_j s_{nj}^4 + 2t\sigma_j^2}) ,$$

since $\frac{n}{\sigma_j^2}(\bar{X}_{nj} - \lambda_j)^2 \sim \chi_1^2(\frac{n(\Theta_j - \lambda_j)^2}{2\sigma_j^2})$. Combining (5.15) - (5.17), we have

$$E_{\underset{\sim}{\Theta}, \underset{\sim}{\Sigma}}[\frac{1}{N^2 \| \underset{\sim}{\bar{X}}_N - \underset{\sim}{\lambda} \|_W^2}] \tag{5.18}$$

$$= E_{\underset{\sim}{\Sigma}}[\frac{1}{N} \int_0^\infty \{ \prod_{j=1}^{p} \left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\right)^{1/2} \} \exp(-tN \Sigma_{j=1}^{p} \frac{(\Theta_j - \lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2})dt] .$$

Next, note that

$$c^{-1} \frac{1}{N} \int_0^\infty \{ \prod_{j=1}^{p} \left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\right)^{1/2} \} \exp(-tN \Sigma_{j=1}^{p} \frac{(\Theta_j - \lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2})dt \tag{5.19}$$

$$= (Nc^{1/2})^{-1}\int_0^{\infty}\{\prod_{j=1}^{p}\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2 c^{1/2}}\right)^{1/2}\}\exp(-t(Nc^{1/2})\Sigma_{j=1}^{p}\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2tc^{1/2}\sigma_j^2})dt\ .$$

As $c \to 0$, $s_{Nj}^4 \xrightarrow{a.s.} \sigma_j^4$ and $Nc^{1/2} \xrightarrow{a.s.} (\text{tr } \underset{\sim}{Q\Sigma})^{1/2}$. Hence, as $c \to 0$,

rhs of (5.19) (5.20)

$$\xrightarrow{a.s.} (\text{tr } \underset{\sim}{Q\Sigma})^{-1/2}\int_0^{\infty}\exp(-t\ (\text{tr } \underset{\sim}{Q\Sigma})^{1/2}\ \Sigma_{j=1}^{p}\frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4})dt$$

$$= (\text{tr } \underset{\sim}{Q\Sigma})^{-1}(\Sigma_{j=1}^{p}\frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4})\ .$$

Next we prove the uniform integrability in $c \le c_0$ of the left hand side of (5.19). With this end, note that,

lhs of (5.19) (5.21)

$$\le \frac{1}{cN}\{\int_0^1 \exp(-tN\Sigma_{j=1}^{p}\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2t\sigma_j^2})dt$$

$$+ \exp(-N\Sigma_{j=1}^{p}\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2\sigma_j^2})\int_1^{\infty}\prod_{j=1}^{p}\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2}\right)^{1/2}dt\}$$

$$\le \frac{1}{cN}\{\frac{1}{N}(\Sigma_{j=1}^{p}\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2\sigma_j^2})^{-1} + \frac{1}{N}(\Sigma_{j=1}^{p}\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2\sigma_j^2})^{-1}$$

$$\cdot \max_{1\le j\le p} (s_{Nj}^4)^{p/2} \int_1^\infty \frac{p}{\prod_{j=1}} \left(\frac{q_j}{2t\sigma_j^2}\right)^{1/2} dt\}$$

$$\le \frac{K}{N^2c} \max_{1\le j\le p} (s_{Nj}^2)^2 + \frac{K}{N^2c} \max_{1\le j\le p} (s_{Nj}^2)^{p+2} ,$$

where $K(>0)$ is a generic constant not depending on c. Thus it suffices to prove the uniform integrability of $(N^2c)^{-1} \max_{1\le j\le p} (s_{Nj}^2)^2$, and $(N^2c)^{-1} \max_{1\le j\le p} (s_{Nj}^2)^{p+2}$ separately. We shall prove in general the uniform integrability of $(N^2c)^{-1} \max_{1\le j\le p} (s_{Nj}^2)^k$, $k \ge 2$ which will handle all of the above terms.

$$E_{\underset{\sim}{\Sigma}}[(N^2c)^{-1} \max_{1\le j\le p} (s_{Nj}^2)^k \, I_{[(N^2c)^{-1} \max_{1\le j\le p} (s_{Nj}^2)^k > d]}] \tag{5.22}$$

$$\le d^{-\delta} E_{\underset{\sim}{\Sigma}}[\{(N^2c)^{-1} \max_{1\le j\le p} (s_{Nj}^2)^k\}^{1+\delta}] ,$$

where $k \ge 2$ and $\delta \in (0, 1/4)$. Taking $\delta' \in (\delta, 1/4)$, and using Holder's inequality, one gets

$$E_{\underset{\sim}{\Sigma}}[\{(N^2c)^{-1} \max_{1\le j\le p} (s_{Nj}^2)^k\}^{1+\delta}] \tag{5.23}$$

$$\le [E_{\underset{\sim}{\Sigma}}(N^2c)^{-(1+\delta')}]^{(1+\delta)/(1+\delta')} [E_{\underset{\sim}{\Sigma}}(\max_{1\le j\le p} s_{Nj}^2)^{k\frac{(1+\delta')(1+\delta)}{\delta'-\delta}}]^{(\delta'-\delta)/(1+\delta')} .$$

Following Ghosh et al. (1976) one gets $P_{\underset{\sim}{\Sigma}}(N \leq \varepsilon n^*) = O(c^{(\frac{1}{2})m})$, for $0 < \varepsilon < 1$. Consequently, following the arguments of the proof of Theorem 2.1, $E_{\underset{\sim}{\Sigma}}[(N^2c)^{-(1+\delta')}]$ can be bounded by a constant independent of c for $m \geq 3$ and $c \leq c_0 < 1$. To bound $E_{\underset{\sim}{\Sigma}}[(\max_{1\leq j\leq p} s_{Nj}^2)^r]$, $r \geq 1$, we need the following two lemmas.

Lemma 5.1 Let $\{U_{nj}, F_n\}$ be a backward submartingale, j=1,...,p. Then $\{\max_{1\leq j\leq p}(U_{nj}), F_n\}$ is a backward submartingale.

Proof: Note that

$$E[\max_{1\leq j\leq p} U_{nj} | F_{n+1}] \geq \max_{1\leq j\leq p} E[U_{nj}|F_{n+1}] \geq \max_{1\leq j\leq p} U_{n+1,j} . \tag{5.24}$$

Lemma 5.2 Let $\{U_n, F_n\}$ be a backward submartingale. Then $\{U_n^r, F_n\}$, $r \geq 1$, is a backward submartingale.

Proof: Note that, by Jensen's inequality,

$$E[U_n^r|F_{n+1}] \geq (E[U_n|F_{n+1}])^r \geq U_{n+1}^r . \tag{5.25}$$

Therefore, by Lemmas 5.1 and 5.2, $(\max_{1\leq j\leq p} s_{nj}^2)^r$, $n \geq m$, is a backward submartingale since s_{nj}^2, j=1,...,p is a backward martingale. Hence, by Doob's maximal inequality for submartingales,

$$E_{\underset{\sim}{\Sigma}}[(\max_{1\le j\le p} s_{Nj}^2)^r] \le E_{\underset{\sim}{\Sigma}}[\sup_{n\ge m}(\max_{1\le j\le p} s_{nj}^2)^r] \tag{5.26}$$

$$\le K_1 E_{\underset{\sim}{\Sigma}}[(\max_{1\le j\le p} s_{mj}^2)^r] ,$$

where K_1 is a constant which may depend on r, m and $\underset{\sim}{\Sigma}$, but not on c. Combining (5.22) - (5.23) and (5.26) the uniform integrability (in $c \le c_o$) of $(N^2c)^{-1} \max_{1\le j\le p} (s_{Nj}^2)^k$, $k \ge 2$, follows. Thus for $m \ge 3$, $c \le c_o < 1$ the uniform integrability of the left hand side of (5.19) follows. Consequently, as $c \to 0$, for $m \ge 3$ and $\underset{\sim}{\theta} \ne \underset{\sim}{\lambda}$,

$$c^{-1}E_{\underset{\sim}{\Sigma}}[\frac{1}{N}\int_0^\infty \{\prod_{j=1}^p (\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2})^{1/2}\}\exp(-tN\Sigma_{j=1}^p \frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2})dt] \tag{5.27}$$

$$\to (\mathrm{tr}\ \underset{\sim\sim}{Q\Sigma})^{-1}(\Sigma_{j=1}^p \frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4})^{-1} .$$

Next, observe that for $\underset{\sim}{\theta} \ne \underset{\sim}{\lambda}$, we have

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|_W^2} \Sigma_{j=1}^p \frac{\sigma_j^2}{s_{Nj}^2}] \tag{5.28}$$

$$= \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}}\left[\frac{1}{n^2\|\bar{\underset{\sim}{X}}_n-\underset{\sim}{\lambda}\|_W^2}\,\Sigma_{j=1}^{p}\,\frac{\sigma_j^2}{s_{nj}^2}\,I_{[N=n]}\right]$$

$$= \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}} E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}}\left[\frac{1}{n^2\|\bar{\underset{\sim}{X}}_n-\underset{\sim}{\lambda}\|_W^2}\,\Sigma_{j=1}^{p}\,\frac{\sigma_j^2}{s_{nj}^2}\,I_{[N=n]}|B_n\right]$$

$$= \Sigma_{n=m}^{\infty} E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}}\left[\frac{1}{n}\,\Sigma_{j=1}^{p}\,\frac{\sigma_j^2}{s_{nj}^2}\,I_{[N=n]}E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}}\left\{\frac{1}{n\|\bar{\underset{\sim}{X}}_n-\underset{\sim}{\lambda}\|_W^2}\mid B_n\right\}\right]$$

$$= E_{\underset{\sim}{\Sigma}}\left[\frac{1}{N}\,\Sigma_{j=1}^{p}\,\frac{\sigma_j^2}{s_{Nj}^2}\int_0^{\infty}\left\{\prod_{j=1}^{p}\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2}\right)^{1/2}\right\}\exp\left(-tN\Sigma_{j=1}^{p}\,\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2t\sigma_j^2}\right)dt\right] .$$

Here, as $c \to 0$,

$$c^{-1}\,\frac{1}{N}\,\Sigma_{j=1}^{p}\,\frac{\sigma_j^2}{s_{Nj}^2}\int_0^{\infty}\left\{\prod_{j=1}^{p}\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2}\right)^{1/2}\right\}\exp\left(-tN\Sigma_{j=1}^{p}\,\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2t\sigma_j^2}\right)dt \tag{5.29}$$

$$\xrightarrow{a.s.} p(\operatorname{tr}\underset{\sim\sim}{Q\Sigma})^{-1/2}\int_0^{\infty}\exp\{-t(\operatorname{tr}\underset{\sim\sim}{Q\Sigma})^{1/2}\,\Sigma_{j=1}^{p}\,\frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4}\}dt$$

$$= p(\operatorname{tr}\underset{\sim\sim}{Q\Sigma})^{-1}\left(\Sigma_{j=1}^{p}\,\frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4}\right)^{-1} ,$$

since as $c \to 0$, $s_{Nj}^2 \xrightarrow{a.s.} \sigma_j^2$, $s_{Nj}^4 \xrightarrow{a.s.} \sigma_j^4$, $j=1,\ldots,p$ and

$Nc^{1/2}$ a.s. $\to (\operatorname{tr} \underset{\sim\sim}{Q\Sigma})^{1/2}$. Next we prove the uniform integrability (in $c \le c_o$) of the left hand side of (5.29). By previous results

$$\text{lhs of (5.29)} \tag{5.30}$$

$$\le \Sigma_{j=1}^{p} \frac{\sigma_j^2}{s_{Nj}^2}\left[\frac{K}{N^2c} \max_{1\le j\le p} (s_{Nj}^2)^2 + \frac{K}{N^2c} \max_{1\le j\le p} (s_{Nj}^2)^{p+2}\right]$$

$$\le (\Sigma_{j=1}^{p} \sigma_j^2) \max_{1\le j\le p} (s_{Nj}^{-2})\left[\frac{K}{N^2c} \max_{1\le j\le p} (s_{Nj}^2)^2 + \frac{K}{N^2c} \max_{1\le j\le p} (s_{Nj}^2)^{p+2}\right].$$

Thus it suffices to prove the uniform integrability of $\max_{1\le j\le p} (s_{Nj}^{-2})(N^2c)^{-1} \max(s_{Nj}^2)^k$, $k \ge 2$, in $c \le c_o$. First we require the following lemma.

Lemma 5.3 Let $\{U_{nj}, F_n\}$ be a backward martingale and let $U_{nj} \ge 0$, for each $j=1,\ldots,p$. Then $\{\max_{1\le j\le p} U_{nj}^{-1}, F_n\}$ is a backward submartingale.

Proof: Note that $f(x) = \frac{1}{x}$ is a convex function for $x > 0$. Therefore, using Jensen's inequality for conditional expectations, we have

$$E[\max_{1\le j\le p} U_{nj}^{-1} | F_{n+1}] \ge \max_{1\le j\le p} E[U_{nj}^{-1} | F_{n+1}] \tag{5.31}$$

$$\ge \max_{1\le j\le p} \{E[U_{nj} | F_{n+1}]\}^{-1}$$

$$= \max_{1\le j\le p} U_{n+1,j}^{-1} .$$

By Lemma 5.3 $\max_{1\le j\le p}(s_{nj}^{-2})$, $n \ge m$, is a backward submartingale since s_{nj}^2, $j=1,\ldots,p$, is a backward martingale. Now, for $\delta \in (0, 1/4)$,

$$E_{\underset{\sim}{\Sigma}}\Big[\max_{1\le j\le p}(s_{Nj}^{-2})(N^2c)^{-1}\Big(\max_{1\le j\le p} s_{Nj}^2\Big)^k \cdot I_{\left[\max_{1\le j\le p}(s_{Nj}^{-2})(N^2c)^{-1}\left(\max_{1\le j\le p} s_{Nj}^2\right)^k > d\right]}\Big] \tag{5.32}$$

$$\le d^{-\delta}E_{\underset{\sim}{\Sigma}}\Big[\Big\{\max_{1\le j\le p}(s_{Nj}^{-2})(N^2c)^{-1}\Big(\max_{1\le j\le p} s_{Nj}^2\Big)^k\Big\}^{1+\delta}\Big] .$$

Using Holder's inequality for $\delta' \in (\delta, 1/4)$,

$$E_{\underset{\sim}{\Sigma}}\Big[\Big\{\max_{1\le j\le p}(s_{Nj}^{-2})(N^2c)^{-1}\Big(\max_{1\le j\le p} s_{Nj}^2\Big)^k\Big\}^{1+\delta}\Big] \tag{5.33}$$

$$\le \Big[E_{\underset{\sim}{\Sigma}}\Big\{\max_{1\le j\le p}(s_{Nj}^{-2})(N^2c)^{-1}\Big\}^{1+\delta'}\Big]^{(1+\delta)/(1+\delta')} \cdot \Big[E_{\underset{\sim}{\Sigma}}\Big\{\max_{1\le j\le p} s_{Nj}^2\Big\}^{k\frac{(1+\delta')(1+\delta)}{\delta'-\delta}}\Big]^{(\delta'-\delta)/(1+\delta')} .$$

Using Holder's inequality again, one gets

$$E_{\underset{\sim}{\Sigma}}\Big[\Big\{\max_{1\le j\le p}(s_{Nj}^{-2})(N^2c)^{-1}\Big\}^{1+\delta'}\Big] \tag{5.34}$$

$$\le \Big[E_{\underset{\sim}{\Sigma}}\Big\{\max_{1\le j\le p}(s_{Nj}^{-2})\Big\}^{2+2\delta'}\Big]^{1/2}\Big[E_{\underset{\sim}{\Sigma}}(N^2c)^{-(2+2\delta')}\Big]^{1/2} .$$

By Lemma 5.2 and Doob's maximal inequality for submartingales

$$E_{\underset{\sim}{\Sigma}}\{\max_{1\le j\le p}(s_{Nj}^{-2})\}^{2+2\delta'} \le E_{\underset{\sim}{\Sigma}}\{\sup_{n\ge m}\max_{1\le j\le p}(s_{nj}^{-2})\}^{2+2\delta'} \tag{5.35}$$

$$\le K_2 E_{\underset{\sim}{\Sigma}}\{\max_{1\le j\le p} s_{mj}^{-2}\}^{2+2\delta'}$$

$$\le K_2 \Sigma_{j=1}^{p} E_{\underset{\sim}{\Sigma}}[s_{mj}^{-2(2+2\delta')}] ,$$

where K_2 is a constant which may depend on m, δ and $\underset{\sim}{\Sigma}$ but not on c. Here, for j=1,...,p and $m \ge 6$,

$$E_{\underset{\sim}{\Sigma}}[s_{mj}^{-2(2+2\delta')}] \tag{5.36}$$

$$= [\sigma_j^2/(n-1)]^{-(2+2\delta')}\, E_{\underset{\sim}{\Sigma}}[(\chi^2_{m-1})^{-(2+2\delta)}]$$

$$= [\sigma_j^2/(n-1)]^{-(2+2\delta')} \int_0^\infty x^{-(2+\delta')} \frac{(1/2)^{\frac{m-1}{2}}\, x^{\frac{m-1}{2}-1} e^{-x/2}}{\Gamma(\frac{m-1}{2})}\, dx$$

$$= [\sigma_j^2/(n-1)]^{-(2+2\delta')} (1/2)^{2+2\delta'}\, \Gamma(\tfrac{m-1}{2} - (2+2\delta'))/\Gamma(\tfrac{m-1}{2})$$

$$< \infty ,$$

since $\delta' \in (\delta, 1/4)$ and hence $\frac{m-1}{2} - (2+2\delta') > 0$. Following the arguments of the proof of Theorem 2.1, using the fact that $P(N \le \varepsilon n^*) = O(c^{(1/2)m})$ for $0<\varepsilon<1$ and that $m \ge 6$ we have $E_{\underset{\sim}{\Sigma}}(N^2 c)^{-(2+2\delta')}$

bounded by a constant independent of c for $c \le c_o < 1$. Also, by previous results $E_{\underset{\sim}{\Sigma}}[\max_{1\le j\le p} s_{Nj}^2]^r$, $r \ge 1$, can be bounded by a constant independent of c. Combining (5.32) - (5.36), the uniform integrability in $c \le c_o$ of the left hand side of (5.29) follows for $m \ge 6$ and $c \le c_o < 1$. Consequently, for $m \ge 6$ and $\underset{\sim}{\theta} \ne \underset{\sim}{\lambda}$, as $c \to 0$,

$$c^{-1}E_{\underset{\sim}{\Sigma}}\Big[\frac{1}{N}\Sigma_{j=1}^p \frac{\sigma_j^2}{s_{Nj}^2}\int_0^\infty \Big\{\frac{p}{\prod_{j=1}^p}\Big(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)^{1/2}\Big\} \tag{5.37}$$

$$\cdot \exp\Big(-tN\Sigma_{j=1}^p \frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)dt\Big]$$

$$\to p\,(\operatorname{tr} Q\underset{\sim}{\Sigma})^{-1}\Big(\Sigma_{j=1}^p \frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4}\Big)^{-1}.$$

Next, observe that for $\underset{\sim}{\theta} \ne \underset{\sim}{\lambda}$,

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\Big[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|_W^2}\Big(\min_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}\Big)\Big] \tag{5.38}$$

$$\le E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\Big[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|_W^4}\Sigma_{j=1}^p \frac{(\bar{X}_{Nj}-\lambda_j)^2}{q_j s_{Nj}^4}\cdot\frac{\sigma_j^2}{s_{Nj}^2}\Big]$$

$$\leq E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}\|_W^2}\left(\max_{1\leq j\leq p}\frac{\sigma_j^2}{s_{Nj}^2}\right)\right].$$

By previous results,

$$E_{\underset{\sim}{\Theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}\|_W^2}\left(\max_{1\leq j\leq p}\frac{\sigma_j^2}{s_{Nj}^2}\right)\right] \tag{5.39}$$

$$= E_{\underset{\sim}{\Sigma}}\left[\frac{1}{N}\left(\max_{1\leq j\leq p}\frac{\sigma_j}{s_{Nj}^2}\right)\int_0^\infty\left\{\prod_{j=1}^p\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2}\right)^{1/2}\right\}\right.$$

$$\left.\cdot\exp\left(-tN\Sigma_{j=1}^p\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2t\sigma_j^2}\right)dt\right].$$

Here, as $c\to 0$, $\max_{1\leq j\leq p}\frac{\sigma_j^2}{s_{Nj}^2}\xrightarrow{a.s.}1$. Hence, as $c\to 0$,

$$c^{-1}\frac{1}{N}\max_{1\leq j\leq p}\frac{\sigma_j^2}{s_{Nj}^2}\int_0^\infty\left\{\prod_{j=1}^p\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2}\right)^{1/2}\right\} \tag{5.40}$$

$$\cdot\exp\left(-tN\Sigma_{j=1}^p\frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4+2t\sigma_j^2}\right)dt$$

$$\xrightarrow{a.s.}(\operatorname{tr}\underset{\sim\sim}{Q\Sigma})^{-1/2}\cdot 1\cdot\int_0^\infty\exp\left\{-t(\operatorname{tr}\underset{\sim\sim}{Q\Sigma})^{1/2}\Sigma_{j=1}^p\frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4}\right\}dt$$

$$= (\operatorname{tr} \underset{\sim\sim}{Q\Sigma})^{-1}(\Sigma_{j=1}^{p} \frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4})^{-1} \ .$$

Also, by previous results, the left hand side of (5.40) has been shown to be uniformly integrable in $c \le c_o$, for $m \ge 6$ and $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$. Consequently, combining (5.39) - (5.40) and the uniform integrability of the left hand side of (5.40), as $c \to 0$, for $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$ and $m \ge 6$,

$$c^{-1}E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}\|_W^2}\ (\max_{1\le j\le p} \frac{\sigma_j^2}{s_{Nj}^2})] \tag{5.41}$$

$$\to (\operatorname{tr} \underset{\sim\sim}{Q\Sigma})^{-1}(\Sigma_{j=1}^{p} \frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4})^{-1} \ .$$

By previous results,

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}\|_W^2}\ (\min_{1\le j\le p} \frac{\sigma_j^2}{s_{Nj}^2})] \tag{5.42}$$

$$= E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}[\frac{1}{N}\ (\min_{1\le j\le p} \frac{\sigma_j^2}{s_{Nj}^2}) \int_0^\infty \{ \prod_{j=1}^{p} (\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2})^{1/2} \}$$

$$\cdot \exp(-tN\Sigma_{j=1}^{p} \frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2})dt] \ .$$

Here, as $c \to 0$, $\min_{1\le j\le p} \frac{\sigma_j^2}{s_{Nj}^2} \xrightarrow{a.s.} 1$. Hence, as $c \to 0$,

$$c^{-1}\frac{1}{N}\Big(\min_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}\Big)\int_0^\infty \Big\{\prod_{j=1}^p \Big(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)^{1/2}\Big\} \tag{5.43}$$

$$\cdot \exp\Big(-tN\Sigma_{j=1}^p \frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)dt$$

$$\xrightarrow{a.s.} (\operatorname{tr} \underset{\sim\sim}{Q}\Sigma)^{-1}\Big(\Sigma_{j=1}^p \frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4}\Big) .$$

Note that with probability one,

$$c^{-1}\frac{1}{N}\Big(\min_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}\Big)\int_0^\infty \Big\{\prod_{j=1}^p \Big(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)^{1/2}\Big\} \tag{5.44}$$

$$\cdot \exp\Big(-tN\Sigma_{j=1}^p \frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)dt$$

$$\le c^{-1}\frac{1}{N}\Big(\max_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}\Big)\int_0^\infty \Big\{\prod_{j=1}^p \Big(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)^{1/2}\Big\}$$

$$\cdot \exp\Big(-tN\Sigma_{j=1}^p \frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)dt ,$$

where the right hand side of (5.44) has been shown to be uniformly

integrable in $c \leq c_0$ for $m \geq 6$ and $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$. Hence the left hand side of (5.44) is uniformly integrable in $c \leq c_0$ for $m \geq 6$ and $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$ so that as $c \to 0$,

$$c^{-1} E_{\underset{\sim}{\Sigma}}\Big[\frac{1}{N}\Big(\min_{1\leq j\leq p} \frac{\sigma_j^2}{s_{Nj}^2}\Big)\int_0^\infty \Big\{ \prod_{j=1}^{p} \Big(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)^{1/2} \Big\} \tag{5.45}$$

$$\cdot \exp\Big(-tN\Sigma_{j=1}^{p} \frac{(\theta_j-\lambda_j)^2}{q_j s_{Nj}^4 + 2t\sigma_j^2}\Big)dt\Big]$$

$$\to (\operatorname{tr} \underset{\sim\sim}{Q\Sigma})^{-1}\Big(\Sigma_{j=1}^{p} \frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4}\Big) .$$

Combining (5.38), (5.41) and (5.45), as $c \to 0$, for $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$ and $m \geq 6$,

$$c^{-1} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\Big[\frac{1}{N^2 \|\bar{X}_N - \underset{\sim}{\lambda}\|_W^4} \Sigma_{j=1}^{p} \frac{(\bar{X}_{Nj}-\lambda_j)^2}{q_j s_{Nj}^4} \cdot \frac{\sigma_j^2}{s_{Nj}^2}\Big] \tag{5.46}$$

$$\to (\operatorname{tr} \underset{\sim\sim}{Q\Sigma})^{-1}\Big(\Sigma_{j=1}^{p} \frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4}\Big)^{-1} .$$

Combining (5.18) - (5.19), (5.27) - (5.29), (5.37) and (5.46) for $m \geq 6$ and $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$, as $c \to 0$,

$$R(\underset{\sim}{\theta},\underset{\sim}{\Sigma},\delta_N^b) = 2c^{1/2} (\operatorname{tr} \underset{\sim\sim}{Q\Sigma})^{1/2} + \frac{c}{2} \frac{\operatorname{tr}(\underset{\sim}{Q}^2\underset{\sim}{\Sigma}^2)}{(\operatorname{tr} \underset{\sim\sim}{Q\Sigma})^2} \tag{5.47}$$

$$+ c\,(\operatorname{tr} \underset{\sim}{Q\Sigma})^{-1}(\Sigma_{j=1}^{p} \frac{(\theta_j-\lambda_j)^2}{q_j\sigma_j^4})^{-1}(b^2-2b(p-2)) + o(c) .$$

For $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$, it follows from (5.18) that

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}[\frac{1}{N^2\|\underset{\sim}{\bar{X}}_N-\underset{\sim}{\lambda}\|_W^2}] \tag{5.48}$$

$$= E_{\underset{\sim}{\Sigma}}[\frac{1}{N}\int_0^\infty \prod_{j=1}^{p} (\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2})^{1/2}\, dt] .$$

Since $Nc^{1/2} \xrightarrow{a.s.} (\operatorname{tr} \underset{\sim}{Q\Sigma})^{1/2}$ and $s_{Nj}^4 \xrightarrow{a.s.} \sigma_j^4$, $j=1,\ldots,p$, then

$$c^{-1/2}\,\frac{1}{N}\int_0^\infty \prod_{j=1}^{p} (\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2})^{1/2}\, dt \tag{5.49}$$

$$\xrightarrow{a.s.} (\operatorname{tr} \underset{\sim}{Q\Sigma})^{-1/2} \int_0^\infty \prod_{j=1}^{p} (\frac{q_j\sigma_j^2}{q_j\sigma_j^2 + 2t})\, dt .$$

Also,

$$c^{-1/2}\,\frac{1}{N}\int_0^\infty \prod_{j=1}^{p} (\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2})^{1/2}\, dt \tag{5.50}$$

$$\le K_3(Nc^{1/2})^{-1} \max_{1\le j\le p} (s_{Nj}^2)^p ,$$

where the right hand side of (5.50) is uniformly integrable in $c \leq c_0$, for $m \geq 2$ (K_3 is a constant independent of c). Using (5.48) - (5.50) one gets as $c \to 0$, for $m \geq 2$ and $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$,

$$c^{-1/2} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2 \| \bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda} \|_W^2}\right] \tag{5.51}$$

$$\to (\operatorname{tr} \underset{\sim}{Q}\underset{\sim}{\Sigma})^{-1/2} \int_0^\infty \prod_{j=1}^{p} \left(\frac{q_j \sigma_j^2}{q_j \sigma_j^2 + 2t}\right) dt \ .$$

Using (5.28), for $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$, we have

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2 \| \bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda} \|_W^2} \sum_{j=1}^{p} \frac{\sigma_j^2}{s_{Nj}^2}\right] \tag{5.52}$$

$$= E_{\underset{\sim}{\Sigma}}\left[\frac{1}{N} \sum_{j=1}^{p} \frac{\sigma_j^2}{s_{Nj}^2} \int_0^\infty \prod_{j=1}^{p} \left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\right) dt\right] .$$

Here, as $c \to 0$,

$$c^{-1/2} \frac{1}{N} \sum_{j=1}^{p} \frac{\sigma_j^2}{s_{Nj}^2} \int_0^\infty \prod_{j=1}^{p} \left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\right)^{1/2} dt \tag{5.53}$$

$$\xrightarrow{a.s.} p(\operatorname{tr} \underset{\sim}{Q}\underset{\sim}{\Sigma})^{-1/2} \int_0^\infty \prod_{j=1}^{p} \left(\frac{q_j \sigma_j^2}{q_j \sigma_j^2 + 2t}\right)^{1/2} dt \ .$$

Further, with probability one,

$$c^{-1/2}\,\frac{1}{N}\sum_{j=1}^{p}\frac{\sigma_j^2}{s_{Nj}^2}\int_0^\infty \prod_{j=1}^{p}\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\right)^{1/2} dt \qquad (5.54)$$

$$\le K_4\,(Nc^{1/2})^{-1}\max_{1\le j\le p}(s_{Nj}^{-2})\max_{1\le j\le p}(s_{Nj}^2)^p\,,$$

where the right hand side of (5.54) is uniformly integrable in $c \le c_0$ for $m \ge 6$ (K_4 is a constant independent of c). Therefore, using (5.52) - (5.54) one gets as $c \to 0$, for $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$ and $m \ge 6$,

$$c^{-1/2}\,E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|_W^2}\sum_{j=1}^{p}\frac{\sigma_j^2}{s_{Nj}^2}\right] \qquad (5.55)$$

$$\to p(\operatorname{tr}\underset{\sim\sim}{Q\Sigma})^{-1/2}\int_0^\infty \prod_{j=1}^{p}\left(\frac{q_j\sigma_j^2}{q_j\sigma_j^2 + 2t}\right)dt\,.$$

Using (5.39) and (5.42) we have for $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$,

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|_W^2}\left(\max_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}\right)\right] \qquad (5.56)$$

$$= E_{\underset{\sim}{\Sigma}}\left[\frac{1}{N}\left(\max_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}\right)\int_0^\infty \prod_{j=1}^{p}\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\right)^{1/2} dt\right]$$

and

$$E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2\|\bar{\underset{\sim}{X}}_N-\underset{\sim}{\lambda}\|_W^2}\left(\min_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}\right)\right] \tag{5.57}$$

$$= E_{\underset{\sim}{\Sigma}}\left[\frac{1}{N}\left(\min_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}\right)\int_0^\infty \prod_{j=1}^p\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2}\right)^{1/2}dt\right].$$

Here, using the fact that $\min_{1\le j\le p}\left(\frac{\sigma_j^2}{s_{Nj}^2}\right) \xrightarrow{a.s.} 1$ and $\max_{1\le j\le p}\left(\frac{\sigma_j^2}{s_{Nj}^2}\right) \xrightarrow{a.s.} 1$ as $c \to 0$, one gets

$$c^{-1/2}\frac{1}{N}\left(\max_{1\le j\le p}\frac{\sigma_j}{s_{Nj}^2}\right)\int_0^\infty \prod_{j=1}^p\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2}\right)^{1/2}dt \tag{5.58}$$

$$\xrightarrow{a.s.} (\operatorname{tr}\underset{\sim\sim}{Q\Sigma})^{-1/2}\int_0^\infty \prod_{j=1}^p\left(\frac{q_j\sigma_j^2}{q_j\sigma_j^2+2t}\right)^{1/2}dt$$

as $c \to 0$, and

$$c^{-1/2}\frac{1}{N}\left(\min_{1\le j\le p}\frac{\sigma_j}{s_{Nj}^2}\right)\int_0^\infty \prod_{j=1}^p\left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4+2t\sigma_j^2}\right)^{1/2}dt \tag{5.59}$$

$$\xrightarrow{a.s.} (\operatorname{tr}\underset{\sim\sim}{Q\Sigma})^{-1/2}\int_0^\infty \prod_{j=1}^p\left(\frac{q_j\sigma_j^2}{q_j\sigma_j^2+2t}\right)^{1/2}dt .$$

Therefore, using the uniform integrability of $(Nc^{1/2})^{-1}\max_{1\le j\le p}\frac{\sigma_j^2}{s_{Nj}^2}$

$\cdot \int_0^\infty \prod_{j=1}^{p} \left(\frac{q_j s_{Nj}^4}{q_j s_{Nj}^4 + 2t\sigma_j^2}\right)^{1/2} dt$ for $m \geq 6$ in $c \leq c_0$, we have by (5.38),

$$c^{-1/2} E_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\left[\frac{1}{N^2 \|\bar{\underset{\sim}{X}}_N - \underset{\sim}{\lambda}\|_W^2} \sum_{j=1}^{p} \frac{(\bar{X}_{Nj} - \lambda_j)^2}{q_j s_{Nj}^4} \frac{\sigma_j^2}{s_{Nj}^2}\right] \tag{5.60}$$

$$+ (\operatorname{tr} \underset{\sim}{Q\Sigma})^{-1/2} \int_0^\infty \prod_{j=1}^{p} \left(\frac{q_j \sigma_j^2}{q_j \sigma_j^2 + 2t}\right)^{1/2} dt \,.$$

as $c \to 0$ when $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$ and $m \geq 6$. Combining (5.51), (5.55) and (5.60) for $m \geq 6$ and $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$, as $c \to 0$,

$$R(\underset{\sim}{\theta},\underset{\sim}{\Sigma},\underset{\sim}{\delta}_N^b) = 2c^{1/2} (\operatorname{tr} \underset{\sim}{Q\Sigma})^{1/2} \tag{5.61}$$

$$+ c^{1/2} (\operatorname{tr} \underset{\sim}{Q\Sigma})^{-1/2} (b^2 - 2b(p-2)) \int_0^\infty \prod_{j=1}^{p} \left(\frac{q_j \sigma_j^2}{q_j \sigma_j^2 + 2t}\right)^{1/2} dt$$

$$+ o(c^{1/2}) \,.$$

From (5.10), (5.47) and (5.61) it follows that

$$R(\underset{\sim}{\theta},\underset{\sim}{\Sigma},\bar{\underset{\sim}{X}}_N) - R(\underset{\sim}{\theta},\underset{\sim}{\Sigma},\underset{\sim}{\delta}_N^b) \tag{5.62}$$

$$= c(\operatorname{tr} \underset{\sim}{Q\Sigma})^{-1} \{\textstyle\sum_{j=1}^{p} (\theta_j - \lambda_j)^2 / (q_j \sigma_j^4)\}^{-1} \, b(2(p-2)-b) + o(c)$$

as $c \to 0$ when $\underset{\sim}{\theta} \neq \underset{\sim}{\lambda}$, and

$$R(\underset{\sim}{\theta},\underset{\sim}{\Sigma},\bar{\underset{\sim}{X}}_N) - R(\underset{\sim}{\theta},\underset{\sim}{\Sigma},\underset{\sim}{\delta}_N^b) \tag{5.63}$$

$$= c^{1/2}\, b\,(2(p-2)-b)\, \mathrm{tr}\,(\underset{\sim\sim}{Q\Sigma})^{-1/2} \int_0^\infty \frac{p}{\prod_{j=1}} \left(\frac{q_j\sigma_j^2}{q_j\sigma_j^2 + 2t}\right)^{1/2} dt + o(c^{1/2})$$

as $c \to 0$ when $\underset{\sim}{\theta} = \underset{\sim}{\lambda}$. The leading term in both (5.62) and (5.63) is maximized when $b = p-2$. In a corresponding fixed sample problem, there is only some numerical evidence in Berger and Bock (1976) that $b = p-1$ may be an optimal choice. The discrepancy between our conclusions needs further exploration.

Remark

As of this time we are unable to show the exact dominance of $\underset{\sim}{\delta}_N^b$ over $\bar{X}_N$. However, the asymptotic results are encouraging.

CHAPTER SIX
SUMMARY AND FUTURE RESEARCH

6.1 Summary

In Chapter 2 we considered a one sample problem with $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$, . . . independent and identically distributed $N(\underset{\sim}{\theta},\sigma^2\underset{\sim}{V})$ where $\underset{\sim}{\theta}$ and σ^2 were unknown and $\underset{\sim}{V}$, a positive definite matrix, was known. The problem was estimation of $\underset{\sim}{\theta}$. The loss was weighted squared error plus cost where the weight matrix, $\underset{\sim}{Q}$, was known and positive definite. In Chapter 3 we considered the Gauss-Markoff set-up with $\underset{\sim}{Y}_n = \underset{\sim}{Z}_n\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}_n$ where $\underset{\sim}{Z}_n$ was a known nxp design matrix, $\underset{\sim}{\beta}$ (px1) was unknown and $\underset{\sim}{\varepsilon}_n$ was distributed $N(\underset{\sim}{0},\sigma^2\underset{\sim}{I}_n)$ where σ^2 was unknown. The problem was estimation of $\underset{\sim}{\beta}$. The loss was again weighted squared error plus cost, the weight matrix being $(\underset{\sim}{Z}_n'\underset{\sim}{Z}_n)$. In Chapter 4 we considered a two sample problem with $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$, . . . independent and identically distributed $N(\underset{\sim}{\theta}_1,\sigma_1^2\underset{\sim}{V}_1)$ and $\underset{\sim}{Y}_1$, $\underset{\sim}{Y}_2$, . . . independent and identically distributed $N(\underset{\sim}{\theta}_2,\sigma_2^2\underset{\sim}{V}_2)$, independently of the $\underset{\sim}{X}_i$'s. Here, $\underset{\sim}{\theta}_i$ and σ_i^2, i=1,2 were unknown while $\underset{\sim}{V}_i$, i=1,2, was a known positive definite matrix. The problem was estimation of $\underset{\sim}{\mu} = \underset{\sim}{\theta}_1 - \underset{\sim}{\theta}_2$. The loss again was weighted squared error plus cost where the weight matrix, $\underset{\sim}{Q}$, was known and positive definite. In Chapter 5 we returned to the one sample problem with $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$, . . . independent and

identically distributed $N(\underset{\sim}{\theta},\underset{\sim}{\Sigma})$ where $\underset{\sim}{\theta}$ and $\underset{\sim}{\Sigma} = \text{diag}(\sigma_1^2,\ldots,\sigma_p^2)$ were unknown. The problem was estimation of $\underset{\sim}{\theta}$. The loss again was weighted squared error plus cost where the weight matrix was $\underset{\sim}{Q} = \text{diag}(q_1,\ldots,q_p)$, $q_j > 0$, $j=1,\ldots,p$.

In all cases full sequential sampling was performed, a class of estimators was shown to dominate the usual estimator (except for Chapter 5), and an asymptotic risk expansion was developed for the above class up to the second order term.

6.2 Future Research

First we should return to the situation in Chapter 5 and try to prove the exact dominance over $\bar{\underset{\sim}{X}}_N$ of the class of estimators we considered. Then we should consider the above problem modified to a two sample situation where we would want to estimate the difference between the two population means. Our goal will be to develop a class of estimators dominating the usual one and produce an asymptotic risk expansion for the above class.

Next we should address the one sample situation with $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$, . . . independent and identically distributed $N(\underset{\sim}{\theta},\underset{\sim}{\Sigma})$ where $\underset{\sim}{\theta} \in R^p$ and $\underset{\sim}{\Sigma}$, positive definite, are both unknown. The problem will be estimation of $\underset{\sim}{\theta}$, the loss would be weighted squared error plus cost where the weight matrix, $\underset{\sim}{Q}$, would be positive definite and the sampling would be fully sequential. The above problem has been addressed in the fixed sample case by Berger and Haff (1983). Then we would consider the above problem modified to a two sample

situation where we would want to estimate the difference between the two population means.

Next, we should return to all the problems considered in this dissertation and the ones mentioned above and do three additional things. The first would be to replace the constant(s) in the James-Stein estimator by a function or functions of the observations which would be bounded between 0 and 2(p-2). This would further broaden the class of estimators, which would include the class of constants, possibly allowing greater risk reduction. The second would be to replace the shrinkage point(s) by the grand mean(s) times a vector of ones. This was first suggested by Lindley (1962) and would perform best when the elements of the mean vector are relatively close to one another. Finally we would like to consider two-stage procedures. The reason for this is that in some situations practical constraints may allow the sampling only to be done in two stages and hence a full sequential sampling scheme would not be feasible.

BIBLIOGRAPHY

Alam, K. (1973). "A Family of Admissible Minimax Estimators of the Mean of a Multivariate Normal Distribution," The Annals of Statistics, 1, 517-525.

Baranchik, A.J. (1970). "A Family of Minimax Estimators of the Mean of a Multivariate Normal Distribution," The Annals of Mathematical Statistics, 41, 642-645.

Berger, J.O. (1976). "Admissible Minimax Estimation of a Multivariate Normal Mean with Arbitrary Quadratic Loss," The Annals of Statistics, 4, 223-226.

Berger, J.O., and Bock, M.E. (1976). "Combining Independent Normal Mean Estimation Problems with Unknown Variances," The Annals of Statistics, 4, 642-648.

Berger, J., and Haff, L.R. (1983). "A Class of Minimax Estimators of a Normal Mean Vector for Arbitrary Quadratic Loss and Unknown Covariance Matrix." Statistics and Decisions, V1, 105-129. Akademische Verlagsgesellschaft, Wiesbaden.

Chou, R., and Hwang, W. (1984). "Sequential Estimation of the Difference Between Two Multivariate Normal Means," Submitted for Publication.

Cressie, N., Davis, A.S., Folks, J.L., and Policello, G.E., II (1981). "The Moment Generating Function and Negative Moments," American Statistician, 35, 148-150.

Efron, B., and Morris, C. (1973). "Stein's Estimation Rule and Its Competitors--An Empirical Bayes Approach," Journal of the American Statistical Association, 68, 117-130.

Efron, B., and Morris, C. (1976). "Families of Minimax Estimators of the Mean of a Multivariate Normal Distribution," The Annals of Statistics, 4, 11-21.

Ghosh, M., and Mukhopadhyay, N. (1976). "On Two Fundamental Problems of Sequential Estimation," Sankhya, Series B, 38, 203-218.

Ghosh, M., and Mukhopadhyay, N. (1980). "Sequential Point Estimation of the Difference of Two Normal Means," The Annals of Statistics, 8, 221-225.

Ghosh, M., and Sen, P.K. (1983-84). "On Two-Stage James-Stein Estimators," Communications in Statistics--Sequential Analysis, 2(4), 359-367.

Ghosh, M., Sinha, B.K., and Mukhopadhyay, N. (1976). "Multivariate Sequential Point Estimation," Journal of Multivariate Statistics, 6, 281-294.

James, W., and Stein, C. (1961). "Estimation with Quadratic Loss." Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, V1, 361-379. University of California Press, Berkeley.

Lindley, D.V. (1962). "Discussion on Professor Stein's Paper," Journal of the Royal Statistical Society, Series B, 24, 285-287.

Mukhopadhyay, N. (1974). "Sequential Estimation of Regression Parameters in Gauss-Markoff Set Up," Journal of the Indian Statistical Association, 12, 39-43.

Rao, C.R. (1965). Linear Statistical Inference and Its Applications. Wiley, New York.

Robbins, H. (1959). "Sequential Estimation of the Mean of a Normal Population," Probability and Statistics, Harold Cramér Volume, 225-245. Almquist and Wiksell, Uppsala.

Starr, N. (1966). "On the Asymptotic Efficiency of a Sequential Procedure for Estimating the Mean," The Annals of Mathematical Statistics, 37, 1173-1185.

Stein, C. (1955). "Inadmissibility of the Usual Estimator for the Mean of a Multivariate Normal Distribution." Proceedings of the 3rd Berkeley Symposium on Mathematical Statistics and Probability, V1, 197-206. University of California Press, Berkeley.

Stein, C. (1981). "Estimation of the Mean of a Multivariate Normal Distribution," The Annals of Statistics, 9, 1135-1151.

Strawderman, W.E. (1971). "Proper Bayes Minimax Estimators of the Multivariate Normal Mean," The Annals of Mathematical Statistics, 42, 385-388.

Takada, Y. (1984). "Inadmissibility of Sequential Estimation Rule of the Mean of a Multivariate Normal Distribution," Sequential Analysis, 3, 267-271.

Woodroofe, M. (1977). "Second Order Approximations for Sequential Point and Interval Estimation," The Annals of Statistics, 5, 984-995.

BIOGRAPHICAL SKETCH

David MacLeod Nickerson was born on June 9, 1958, in Memphis, Tennessee. In 1960 his family moved to Orlando, Florida, and he eventually attended the University of Central Florida where he received a Bachelor of Science (1980) degree in mathematics. In August of 1980 he moved to Gainesville, Florida, to pursue a master's degree in statistics at the University of Florida. After he received his degree in 1982 he chose to continue on towards a Ph.D. in statistics at the University of Florida. During his time at the University of Florida, David was employed as a graduate research assistant, performing statistical consulting duties at the Institute of Food and Agricultural Sciences Statistics Unit and the Biostatistics Unit. He was also employed as a teaching assistant in the Department of Statistics. Upon graduation, he will join the Biostatistics Unit of the Department of Statistics at the University of Florida as an assistant research scientist.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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